

ELEMENTARY STATICS OF TWO AND THREE DIMENSIONS



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R. J. A. BARNARD

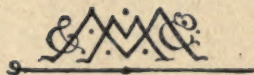






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**ELEMENTARY STATICS
OF TWO AND THREE DIMENSIONS**



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ELEMENTARY STATICS

OF TWO AND THREE DIMENSIONS

BY

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PREFACE

THE reception of my *Elementary Dynamics* has encouraged me to write a *Statics* of a similar standard of difficulty, and I have followed the same principles as in that book.

With reference to the contents I may mention the following :

- (i) The introduction of examples on friction early, in dealing with forces at a point.
- (ii) The insertion of a chapter on Shearing Stress and Bending Moments.
- (iii) The use of the methods of the elementary differential calculus in the chapter on Virtual Work, and of the integral calculus for finding the centres of mass of a number of geometrical areas and volumes.
- (iv) Three-Dimensional Statics are dealt with more fully than usual.
- (v) A chapter on Vectors in Space, including vector products, has been included.

It is hoped that the last-named chapter, with the chapter on vectors in my *Dynamics*, will make a satisfactory introduction to Vector Algebra and its applications.

As in my *Dynamics*, a number of the more difficult examples come from papers set at the Melbourne University. Nearly all the others have been constructed by myself for the book or for my classes.

R. J. A. BARNARD.

DUNTRON,
March, 1921.

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CHAPTER I.

FOUNDATIONS OF MECHANICS.

BEFORE dealing specially with Statics as one branch of Mechanics, it is necessary to refer to the fundamental ideas about Force, and these are most easily obtained from a consideration of dynamical questions.

It will be sufficient to give a short summary of these ideas, referring for fuller development to my book on *Elementary Dynamics*, Chapter III.

1. Position, Relative Motion and Rest.

The position of a point is generally measured by its distances from points, or lines, or planes fixed to the earth. If these distances change, it is said to be in motion relative to the earth; if the distances do not change it is at rest relative to the earth. When it is at rest relative to the earth it is usually said simply to be at rest.

2. Force.

Force is generally defined as any action that alters or tends to alter a body's state of motion; but two or more forces acting on a body may balance one another's effects in such a way that the body remains at rest.

Statics is that branch of mechanics which deals with the relations between forces acting on a body at rest.

A body is said to be **Rigid** if its shape and size are unaffected by forces acting on it. While no such body actually exists in nature, the alterations in solids are small and will be neglected throughout this book, except where changes are specially mentioned, such as in the springs and strings of the next paragraphs.

In simple experiments a force may be applied to a body by attaching a spring or string to it, and pulling the spring or string. The effect of the force depends both on the strength and the direction of the pull. Hence a force requires both magnitude and direction to denote it fully, and can be consequently represented by a straight line whose length represents the magnitude of the force according to some scale, and whose direction is the direction of the force.

If the force is exerted by the medium of a spiral spring (spring balance), the extension of the spring can be taken as a measure of the force. For, if we take weights which balance on an ordinary balance, and hang them successively to a spring balance, we find that they elongate the spring balance to the same amount; and that if different weights are hung on, the elongation will be very nearly proportional to the weight, and even if the elongation is not exactly proportional to the weight, the balance may still be graduated to shew various forces by marking the positions taken by the pointer for given weights hung on the balance.

Spring balances thus graduated, and of comparatively small stiffness, so that a weight of a few ounces produces an easily observed elongation, may be used for simple demonstrations of fundamental results in statics.

3. Action and Reaction.

A force acting on a body is always due to the presence of a second body influencing it, and a force never acts on one body without an equal force acting in the opposite direction

on the second. This fact is called Newton's **Law of Action and Reaction**.

For example, when a heavy body rests on a horizontal table, the body presses down on the table, and the table presses upwards on the body with exactly the same force. A verification of this law in certain cases may be made by means of the spring balances. If two of these are hooked together and then pulled, it will be found that the two register the same force, shewing that they exert the same force to one another. As a general rule, the bodies acting and re-acting on one another are in contact, as in the above cases, but all bodies also act on one another by gravitational attraction, and in this case the two reacting bodies pull one another without contact. This force, however, is only appreciable in ordinary statics when one of the bodies is the earth, the gravitational attraction between two bodies on the earth being always very minute.

4. Weight.

The attraction which the earth exerts on a body results in the body having weight.

Weight is the force with which a body presses on a horizontal plane on which it is at rest, or the force with which a body at rest, and supported at one point (for example by a string), will exert on its support.

On account of the rotation of the earth, this is slightly less than the force with which the earth attracts the body (see *Elementary Dynamics*, Art. 130). However, it is not necessary in Statics to consider the rotation of the earth, if we replace the attraction of the earth by a slightly different force equal to the weight of the body; in other words, if we think of the body as acted on by a vertical downward force equal to its weight. This force, and the other forces acting on the body, will balance one another in the sense referred to above (Art. 2), as if there were no rotation of the earth.

5. Transmissibility of Force.

The principle called the Transmissibility of Force states that a force may be supposed to act at any point in its line of action, or in other words, the effect on a body is the same at whatever point in the line of action the force is applied to the body.

Thus, if we have a block of wood on a horizontal table, it will take the same force to move it, whether we pull it from the front or push it from behind, provided the line of action of the pull and push is the same.

6. Compounding of Two Forces.

Besides the Laws of Action and Reaction, and of the Transmissibility of Force, Statics requires another experimental basis to rest on. This is the rule or rules concerning the combined action of two forces meeting at one point.

The facts may be expressed in two statements :

(I) If two forces P and Q act along the same line in the same direction, they are equivalent in their effect to a single force equal to their sum, and if they act in opposite directions they are equivalent to a single force equal to their difference, in the direction of the larger.

In particular, if two equal forces act on a body in opposite directions, along the same line, the body can remain at rest ; and conversely, if a body remains at rest under the action of two forces only, the forces must be equal, opposite, and in the same straight line.

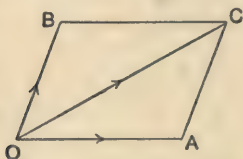


FIG. 1.—Parallelogram of forces.

(II) The Parallelogram of Forces.

If two forces not in the same straight line act through one point O in a body, and are represented in direction and magnitude by lines OA and OB , they are equivalent in their effect to a single force represented by OC ,

the diagonal of the parallelogram of which OA and OB are adjacent sides.

The first of these statements, relating to forces in the same line, may be regarded as a special case of the parallelogram of forces.

Experimental verifications of these statements may be made with the help of the spring balances referred to already.

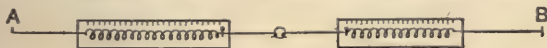


FIG. 2.—Equilibrium of two forces.

For example, first attach two spring balances to a small ring and pull them. The other ends may be fixed to two points A and B . It will be found when the ring is at rest that the spring balances are pulling in opposite directions, and the pointers shew that the same force is exerted by or on each.

The other part of the first statement may then be verified by having two spring balances pulling side by side in the same direction, and one pulling in the opposite direction.

To verify the parallelogram of forces, it is only necessary to modify the last arrangement by fixing three springs at one end of each to three points A , B , C , in a plane, say on a sheet of paper pinned to a drawing board.



FIG. 3.—Parallelogram of forces.

Fix the other ends to a small ring as before. When the ring comes to rest mark off lengths OH , OK , OL , proportional to the three pulls P , Q , R , shewn by the springs, and draw the

parallelogram OHMK. Then it will be found that OM is in the same line as OL, and equal to OL. This shews that since R balances a force equivalent to P and Q, therefore the forces P and Q represented by OH and OK are together equivalent to a force represented by OL.

7. Tension of a String.

If a string is pulled from two ends and remains at rest, the forces at the two ends will be the same, provided the weight of the string is negligible, for the string is then acted on by only two forces which must be equal and opposite.

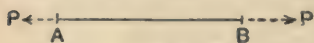


FIG. 4.



FIG. 5.

Instead of the whole string, we may take any portion AC, and say that for the same reason the portion AC is pulled by equal forces at the two ends. In other words, at C the portion BC is pulling at the portion AC with a force P, and for the same reason the portion AC is pulling at the portion CB with the same force P. So that at any point C of the string we have the portion on either side of C pulling at the other portion, and these pulls are the same at all points along the string. This force P is called the *Tension of the String*.



FIG. 6.—Apparatus for parallelogram of forces.

The same result applies to a weightless string passing round a frictionless peg or pulley.

Thus, in the verification of the parallelogram of forces the strings may be replaced by strings passing over smooth pulleys and having weights attached at the ends, as in Fig. 6.

The pulleys are in a vertical plane, and if the strings are knotted at O or attached to a small ring there, we have three forces P, Q, R, acting at O in the directions shewn, and the parallelogram of forces may be verified as before.

8. Units of Force.

In all the Statics it is only ratios of the forces that come into the equations, hence there are no difficulties about units such as appear in the Dynamics ; consequently, in Statics the forces may be measured in any unit so long as the same unit is retained throughout the one question. We will use the Pound-weight, Gram-weight, or Kilogram-weight as our usual units.

CHAPTER II.

FORCES AT A POINT.

9. Two Forces.

We have seen that when a body is acted on by two forces only and at rest, these forces must be equal, opposite, and in the same straight line.

In such a case it is said that the body is in equilibrium, or that the forces are in equilibrium. Both expressions are in common use.

10. Three Forces.

When three forces act on a body and meet at a point and are in equilibrium, we can get the relations between the forces by the parallelogram of forces. For if P , Q , R , in Fig. 7 are

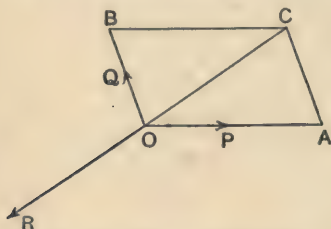


FIG. 7.—Equilibrium of three forces.

the three forces, and two of them are represented in direction and magnitude by OA and OB , the resultant of these two must be represented by OC in the figure, and must balance R , or R can be represented in direction and magnitude by CO .

Hence we notice that the three forces P , Q , R , may be represented in direction and magnitude by the three sides OA , AC , CO , of a triangle. Notice, however, that AC represents Q in magnitude and direction, but not in position.

We thus get the proposition known as the **Triangle of Forces**, namely :

If three forces acting at a point can be represented in direction and magnitude by the three sides of a triangle taken in order (that is, in order round the triangle as OA, AC, CO), they are in equilibrium ; and conversely,—

If three forces acting at a point are in equilibrium, they can be represented by the three sides of a triangle taken in order.

It will be noticed that the three lines of action of the forces must be in one plane ; also that if three forces in one plane are in equilibrium, they must all meet in a point or be all parallel.

For if any two of them, say P and Q, meet in a point O, the resultant of P and Q acts through O, and can only be balanced by a single force if this force acts through O.

We shall find later that if two forces are parallel, they can be balanced by a third force parallel to these and in the same plane ; also that three forces, which are not in the same plane, cannot balance one another.

11. Instead of using the Triangle of Forces we can use other methods of dealing with three forces. The same theorem expressed in trigonometrical form is often called **Lami's Theorem**, and states that

If three forces acting at a point are in equilibrium, each is proportional to the sine of the angle between the other two.

For if in Fig. 8 α is the angle between Q and R,

β is the angle between R and P,

γ is the angle between P and Q,

$$\angle COA = 180 - \beta,$$

$$\angle OCA = 180 - \alpha,$$

$$\angle OAC = 180 - \gamma ;$$

also
$$\frac{P}{OA} = \frac{Q}{AC} = \frac{R}{CO} ;$$

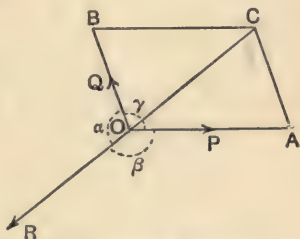


FIG. 8.—Lami's theorem.

and by trigonometry,

$$\frac{OA}{\sin(180 - \alpha)} = \frac{AC}{\sin(180 - \beta)} = \frac{CO}{\sin(180 - \gamma)},$$

$$\therefore \frac{OA}{\sin \alpha} = \frac{AC}{\sin \beta} = \frac{CO}{\sin \gamma};$$

$$\therefore \frac{P}{\sin \alpha} = \frac{Q}{\sin \beta} = \frac{R}{\sin \gamma}.$$

Examples. 1. A particle P, of weight 4 lbs., is supported by a string attached to a fixed point O, and is pulled from the vertical by a horizontal force of 3 lbs.-wt.; find the angle the string will make with the vertical, and the tension of the string.

Let T be tension in lbs.-wt. Draw PN horizontal and ON vertical. Then ONP is a triangle of forces, for the force 4 is parallel to ON, 3 is along NP, and T is along PO. Therefore the forces are proportional to the sides of this triangle.

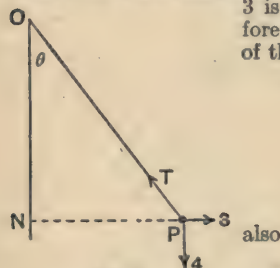


FIG. 9.

$$\therefore \frac{PN}{ON} = \frac{3}{4};$$

$$\therefore \tan \theta = \frac{3}{4};$$

$$\therefore \theta = 36^\circ 52';$$

$$\frac{T}{4} = \frac{PO}{ON} = \frac{5}{4};$$

$$\therefore T = 5 \text{ lbs.-wt.}$$

2. The ends of a string 17 inches long are attached to two points 13 inches apart in the same horizontal, and a weight of 4 lbs. is attached to the string 5 inches from one end. Find the tension in each portion of the string.

At the point C, where the weight is attached, there are three forces acting; the two portions of the string are pulling at the knot with forces which we have denoted in the figure by T_1 and T_2 , and the weight is pulling with a force of 4 lbs.

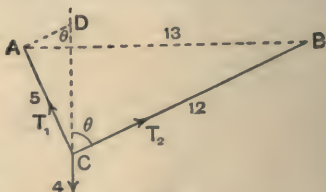


FIG. 10.

Since the sides of ABC are 5, 12, and 13 inches, the triangle is right angled at C, and if CA makes an angle θ with the horizontal, CB makes an angle θ with the vertical.

We can now proceed in two ways.

First solution, by the Triangle of Forces.

To get a convenient triangle with its sides parallel to the three forces, draw AD parallel to CB, and CD vertical; then ADC has its sides parallel to the three forces, following round the triangle in the direction ADCA. The forces T_1 , T_2 , and 4 are therefore proportional to CA, AD, DC.

Now the triangle ADC is similar to CAB.

$$\therefore \frac{CA}{12} = \frac{AD}{5} = \frac{DC}{13};$$

$$\therefore \frac{T_1}{12} = \frac{T_2}{5} = \frac{4}{13};$$

$$\therefore T_1 = \frac{48}{13} = 3.69 \text{ lbs.-wt.}$$

and

$$T_2 = \frac{20}{13} = 1.54 \text{ lbs.-wt.}$$

Second solution, by Lami's Theorem.

The angles between the forces are

180 - θ between T_1 and 4,

90 + θ between T_1 and 4,

90 between T_1 and T_2 ,

$$\frac{T_1}{\sin(180 - \theta)} = \frac{T_2}{\sin(90 + \theta)} = \frac{4}{\sin 90},$$

$$\frac{T_1}{\sin \theta} = \frac{T_2}{\cos \theta} = \frac{4}{1};$$

but

$$\sin \theta = \frac{12}{13} \quad \text{and} \quad \cos \theta = \frac{5}{13},$$

$$T_1 = \frac{48}{13} \text{ lbs.-wt.}, \quad T_2 = \frac{20}{13} \text{ lbs.-wt.}$$

3. A string of length 24 inches has its ends fixed to two points 20 inches apart in the same horizontal line, and a weight of 6 lbs. is hung from it by means of a small hook. Find the tension of the string. (Neglect friction; see Art. 18.)

The difference between this and Example 2 is that the hook can slip along the string, and the two portions of the string form one continuous string with the same tension throughout. In Example 2, the two portions being knotted to the third were equivalent to two quite distinct strings with different tensions.

As the tensions are the same, either Lami's Theorem or the parallelogram of forces shews that the two portions make equal

angles with the vertical, and therefore the weight hangs from the middle point of the string. The solution can then be completed as in Example 2.

Thus, in the figure CAD is a triangle of forces.

$$\therefore \frac{T}{12} = \frac{6}{2\sqrt{44}};$$

$$\therefore T = \frac{18}{\sqrt{11}} = 5.43 \text{ lbs.-wt.}$$

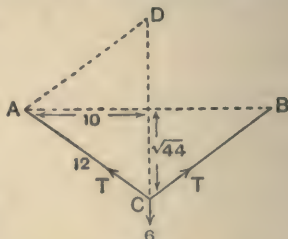


FIG. 11.

4. A weight of 10 lbs. is supported by two strings, which make angles of 30° and 60° with the vertical. What are the tensions of the strings?

5. A weight of 20 lbs. is supported by two strings of lengths 3 and 4 feet, to two points 5 feet apart in the same horizontal. Find the tensions of the strings.

6. A weight of 20 lbs. is supported by a string AB attached to a fixed point A, and is pulled aside from the vertical by a horizontal force. If AB makes an angle 30° with the vertical in its final position, find the horizontal force and the tension of the string.

7. Solve Example 6, with the angle 30° changed to 20° .

8. If a weight of 10 lbs. is supported by a string 5 feet long, how far will it be pulled from the vertical by a horizontal force of 3 lbs. applied to it? Find also the tension of the string in the final position, and the amount the weight has been raised by the action of the force.

9. A weight W is supported by two strings at right angles to one another, and attached to points in the same horizontal line; prove that their tensions are inversely proportional to their lengths.

10. A body of weight 12 lbs. is supported by two strings, which make angles 40° and 70° with the vertical. Find their tensions.

11. A picture is supported symmetrically in a vertical plane by a wire passing over a nail. If the wire is attached to fixed points A, B, at the same level in the picture, shew that the shorter the length of the wire the greater will be its tension.

12. A picture weighing 5 lbs. is suspended symmetrically by a wire 26 inches long passing over a nail 5 inches above the line joining the points on the picture to which the wire is attached. Find its tension.

13. A weight of 2 lbs. is hung by a ring from a string passing over two smooth pegs in the same horizontal line, and supporting

weights of $1\frac{1}{2}$ lbs. at each end. Find the angles the inclined portions of the string make with the vertical.

Will the result be affected if one of the pegs is placed at a higher level than the other ?

14. A string passes over two smooth pegs on the same level and supports weights of 5 lbs. at each end. A weight of 6 lbs. is now hung to it midway between the pegs, and the point of attachment of this weight moves down 3 inches. Find the distance between the pulleys, and find the depth of the middle point below them when any weight W is hung on.

Examine the case when $W = 8$. What happens if $W = 10$ lbs. ?

15. If three forces, of magnitudes 3, 4, 5 lbs.-wt., are in equilibrium, find by the triangle of forces the angles between each pair.

16. If three forces, of magnitudes 5, 6, 7 lbs.-wt., are in equilibrium, find by the triangle of forces the angles between each pair.

12. Resultant and Components.

The force R , which is equivalent in its action to two given forces P and Q , is called their **Resultant**, and P and Q are called the **Components** of R .

A force can be replaced by two components in an infinite number of ways ; thus, if OC represents the force, and Ox , Oy are any two lines through O , it is only necessary to draw CA , CB parallel to Oy and Ox to get the components OA , OB in these two directions.

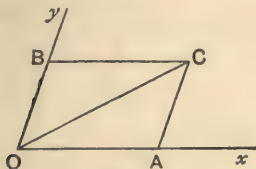


FIG. 12.

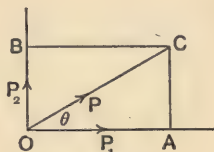


FIG. 13.

The most important case of components is where the components are at right angles ; the components are then usually called **Resolved Parts or Resolutes** of the force.

Let P be the resultant force (in Fig. 13), and suppose it is required to find its resolutes in two directions, Ox and Oy at right angles, where Ox makes an angle θ with P .

Let P_1 and P_2 be the resolutes. Then, in the figure,

$$\frac{P_1}{P} = \frac{OA}{OC} = \cos \theta,$$

$$\frac{P_2}{P} = \frac{OB}{OC} = \frac{AC}{OC} = \sin \theta;$$

$$\therefore P_1 = P \cos \theta, \dots\dots\dots(1)$$

$$P_2 = P \sin \theta; \dots\dots\dots(2)$$

also

$$OA^2 + OB^2 = OC^2;$$

$$\therefore P_1^2 + P_2^2 = P^2 \dots\dots\dots(3)$$

and

$$\frac{P_2}{P_1} = \frac{AC}{OA} = \tan \theta. \dots\dots\dots(4)$$

(1) and (2) give the resolutes in two directions at right angles when the magnitude and direction of the resultant are given; (3) and (4) give the magnitude and direction of the resultant when the resolutes in two directions at right angles are given.

13. When the components make any angle α with one another, the equations are rather more complicated. Using other letters as before, we have from Fig. 14,

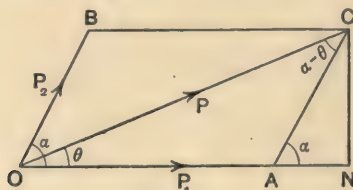


FIG. 14.

$$P^2 = P_1^2 + P_2^2 + 2P_1P_2 \cos \alpha, \dots\dots\dots(5)$$

$$\tan \theta = \frac{P_2 \sin \alpha}{P_1 + P_2 \cos \alpha}, \dots\dots\dots(6)$$

$$P_1 = P \sin (\alpha - \theta) / \sin \alpha, \dots\dots\dots(7)$$

$$P_2 = P \sin \theta / \sin \alpha, \dots\dots\dots(8)$$

all following from elementary trigonometry. They should be verified by the student.

Example. Shew that equations (5)-(8) reduce to (1)-(4) if α is a right angle.

14. It is important to observe that *the resolute of a force in any direction is obtained by multiplying the force by the cosine of the angle between it and the given direction.*

The following theorem is very important :

The algebraic sum of the resolutes of two forces in a given direction is equal to the resolute of the resultant in the same direction.

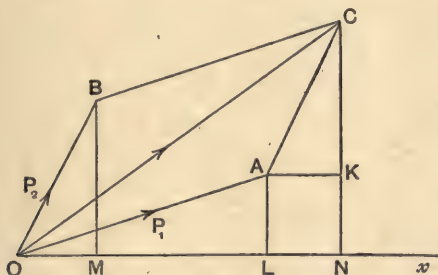


FIG. 15.

For, in Fig. 15, OL is the resolute in the direction Ox of P_1 ; $LN = AK = OM$, and is the resolute of P_2 ; while ON is the resolute of P in the same direction, and

$$ON = OL + LN = OL + OM,$$

which proves the theorem.

Examples. **1.** Find the resolutes of a force of 50 lbs.-wt. in two directions, making angles 40° and 50° with the force on opposite sides of it.

2. Find the magnitude and direction of the resultant of two forces, each of 6 lbs., at an angle of 120° with one another.

3. Find the magnitude and direction of the resultant of two forces of respectively 6 and 8 lbs.-wt., which make an angle 60° with one another.

15. More than Two Forces.

To consider the effect of more than two forces we can proceed by steps. Thus, if OA , OB , OC , OD , represent four forces acting at a point, OA and OB can be combined into a

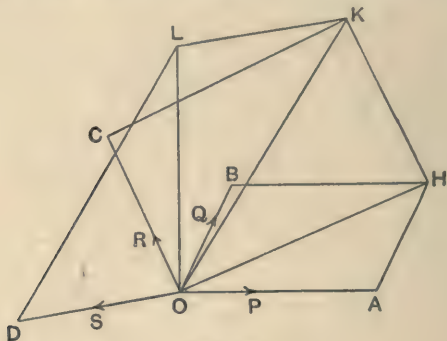


FIG. 16.

single force OH , then OH and OC into a single force OK , and finally OK and OD into a single force OL , and so on for any number of forces.

It is easy to see that the whole parallelograms in the figure need not be drawn. All that is necessary to get the resultant graphically, is to draw from O a line OA to represent P , then from A , AH representing Q , HK representing R , and KL representing S . Then OL will represent the resultant of P , Q , R , S .

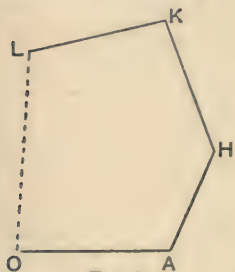


FIG. 17.

If the last point coincides with O there is no resultant, or the forces are in equilibrium, and we thus get the result known as the **polygon of forces**, namely,

If a number of forces acting at a point can be represented in direction and magnitude by the sides of a polygon taken in

order, the forces are in equilibrium; and conversely, if they are in equilibrium, they can be represented by the sides of a polygon taken in order.

Notice, however, that when there are more than three forces whose directions are known, the relative magnitudes cannot be deduced by the polygon of forces, as it can when there are only three forces, for different polygons may have their sides in given directions and yet not be similar, as, for example, a square and a rectangle.

Notice, also, that the polygon of forces applies even if the forces are not all in one plane. For example, if ABCD represents a tetrahedron, forces acting at a point and represented in direction and magnitude by AB, BD, DC, CA would be in equilibrium.

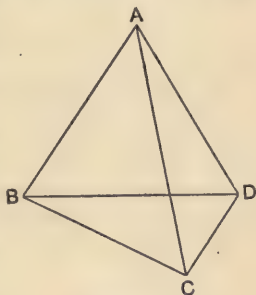


FIG. 18.

The figure ABDCA is called a Skew Quadrilateral, its sides not being all in one plane.

16. *Algebraic method of finding the resultant of a number of concurrent coplanar forces.*

The best way of dealing with a number of forces is to apply

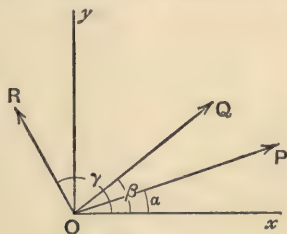


FIG. 19.

the result of Art. 14. This will be true for any number of forces. Hence, if P, Q, R, ... etc., are any forces acting at O,

and making angles $\alpha, \beta, \gamma, \dots$ etc., with a line Ox , and if the resultant is F , making an angle θ with Ox , and if X, Y are the resolutes of F along and perpendicular to Ox , then

$$X = F \cos \theta = P \cos \alpha + Q \cos \beta + R \cos \gamma + \dots$$

$$\text{and} \quad Y = F \sin \theta = P \sin \alpha + Q \sin \beta + R \sin \gamma + \dots,$$

from which F and θ can be found.

If the forces are in equilibrium,

$$F = 0;$$

$\therefore$ both X and Y are zero.

17. Conditions for Equilibrium.

We can now state the conditions for equilibrium of a number of concurrent coplanar forces, thus :

The forces are in equilibrium if the sums of the resolutes in any two different directions separately vanish.

For if the sum of the resolutes in a direction Ox is zero,

$$F \cos \theta = 0,$$

and either

$$F = 0$$

or

$$\theta = 90^\circ,$$

that is to say, if the resultant exists, it is at right angles to Ox . Similarly, if the sum of the resolutes in another direction Oy is zero, the resultant, if existing, is perpendicular to Oy .

Hence, as F cannot be perpendicular to two different lines, it must be zero if both these sums are zero.

It is seen from this that it is not necessary for Ox and Oy to be at right angles, but it is generally most convenient to choose directions at right angles.

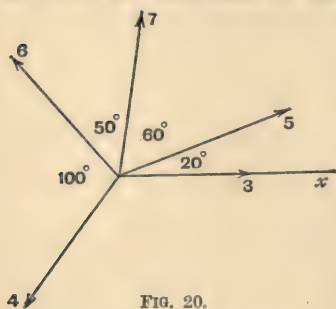


FIG. 20.

Examples. 1. Find the resultant of the forces marked in the figure.

It is convenient to tabulate the work thus :

Force.	Angle with Ox.	cos.	sin.	Resolute along Ox.	Resolute along Oy.
3	0°	1.0	0.0	3.0	0.0
5	20°	0.9397	0.3420	4.6985	1.7100
7	80°	0.1736	0.9848	1.2152	6.8936
6	130°	-0.6428	0.7660	-3.8568	4.5960
4	230°	-0.6428	-0.7660	-2.5712	-3.0640
Total				2.4857	10.1356

Having found the sums of the resolutes X and Y along Ox and Oy, it is best to find the angle θ first, and then R by the equations

$$\tan \theta = \frac{Y}{X},$$

$$\frac{R}{X} = \sec \theta,$$

it being more convenient to find R in this way, using logarithms, than by the equation

$$R = \sqrt{X^2 + Y^2},$$

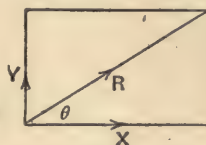


FIG. 21.

unless tables of squares are available.

Notice also in finding θ to be careful which quadrant the resultant is in. This can only be settled by thinking of the signs of X and Y separately. In the above example, for instance,

$$\tan \theta = \frac{10.1356}{2.4857};$$

$$\therefore \theta = 76^\circ 13',$$

θ being in the first quadrant (not the third), because both X and Y are positive.

$$R = X \sec \theta = 10.43.$$

2. A body of weight 6 lbs. is supported by two strings, one of which makes an angle 60° with the vertical, and exerts a pull of 8 lbs.-wt. What is the direction and pull of the other ?

3. Two forces at right angles, one of which is 5 lbs., balance a third force of 6 lbs. Find the magnitude of the second force and the direction of the third.

4. Find the resultant in direction and magnitude of the four forces in Fig. 22.

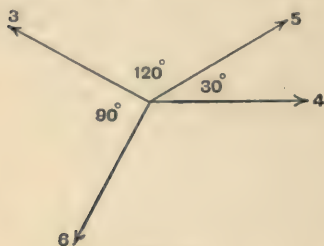


FIG. 22

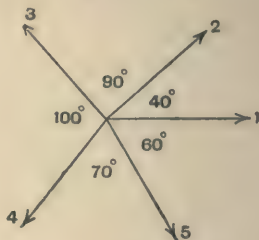


FIG. 23.

5. Find the direction and magnitude of the resultant of the forces in the diagram (Fig. 23).

6. Forces act as in Fig. 24. If they are in equilibrium, find P and Q .

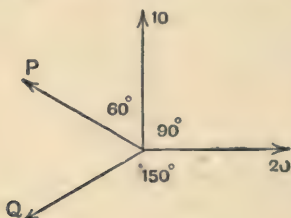


FIG. 24.

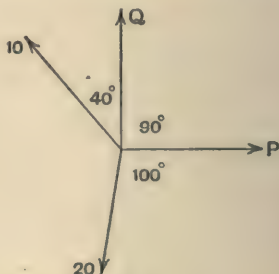


FIG. 25.

7. Forces act as in Fig. 25 and are in equilibrium. Find P and Q .

18. Friction.

Friction is one of the most frequently occurring forces in Nature. Whenever two bodies are in contact, and forces are acting on them tending to make one move the other, friction comes into play in a direction tending to oppose the motion.

The principal facts about friction may be learnt by the use of simple apparatus consisting of a plane which can be placed

horizontal or inclined at any angle, and weights, etc., as shewn in the diagrams.

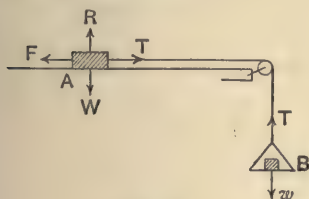


FIG. 26.

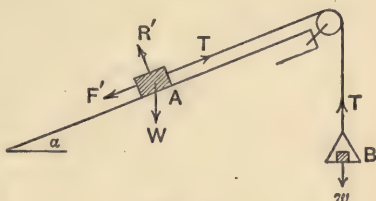


FIG. 27.

A represents a weight to which a string, passing over a pulley and carrying a scalepan B, is attached. In the diagrams the scalepan and weights in it are treated as one body, and all the forces acting on A and B are marked. The pressure exerted on W by the plane has been replaced by two resolved parts, R in Fig. 26, R' in Fig. 27, perpendicular to the plane, and F in Fig. 26, F' in Fig. 27, parallel to the plane. R (or R') is called the normal reaction, F (or F') is the friction. The tension of the string T is shewn in the way it acts on A and also as it acts on B; w is the combined weight of scalepan and weights in it.

If weights are gradually added to the scalepan, A does not at first move, shewing that the pull of the string causes friction to be brought into play in the opposite direction, which balances this pull. But if the weights in B are increased, a stage will be reached when the bodies move, shewing that the friction is no longer able to balance the tension. Thus we find that there is a maximum value to the friction that can be brought into play in any case.

So long as the bodies do not move, we have from the general equations of equilibrium in Fig. 26,

$$w = T = F;$$

but in Fig. 27,

$$w = T = F' + W \sin \alpha,$$

for the sum of the resolutes along the plane of the forces

acting on A is zero. Thus, by increasing the weight w till the bodies just move, we can deduce the maximum friction in either case.

Also, in Fig. 26, $R = W$;

but in Fig. 27, taking resolutes perpendicular to the plane,

$$R' = W \cos \alpha.$$

If we continue the experiments, varying the weight of A by placing other weights on top of it, we find that the maximum friction is nearly proportional to the normal reaction, and that the ratio

$$\frac{\text{maximum friction}}{\text{normal reaction}}$$

is the same on the horizontal plane or the incline, and depends only on the nature of the surfaces in contact (including in nature, degree of polish), and not on the area or shape of the surfaces in contact.

We will therefore take it as a closely approximate result that

$$\frac{F_{\max}}{R} = \mu,$$

where $F_{\max}$ denotes the maximum friction and μ is a constant for a given pair of bodies in contact.

μ will be called the **coefficient of friction**.

The value of μ varies greatly for different pairs of materials. For smooth blocks of wood μ may lie between 0.25 and 0.50. For wood on glass, between 0.20 and 0.40. But there is no limit to the values μ may assume in different cases; it may have any value from zero upwards.

The beginner must be careful to notice that the coefficient of friction is the ratio of the maximum friction to the *normal component* of the *reaction*, *not* of the maximum friction to the *weight*. The normal reaction may be quite different from the weight. Also it must always be remembered that when a certain value is given for the coefficient of friction, while the friction cannot be greater than μ times the normal reaction,

it may according to circumstances be anything less than that maximum.

When the surfaces are such that no friction can act between them, or $\mu=0$, they are said to be perfectly smooth. This never occurs in nature. However, in many examples bodies will be spoken of as smooth, meaning that no friction is supposed to act. It does not necessarily imply that the results would be affected if friction could act. For example, if we are asked what force acting in a certain direction is required to keep a given body from sliding down a given smooth inclined plane, we know that such a plane does not exist, but we also know that if we calculate the force on the supposition that there is no friction, then if that force is applied the body will rest on the plane without any friction being brought into play.

19. Angle of Friction.

When the maximum friction is acting, the resultant reaction S (compounded of the normal reaction and the friction) makes a maximum angle with the normal.

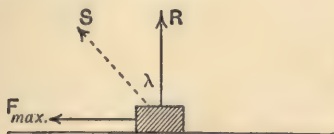


FIG. 28.

For the angle θ between S and R is given by

$$\tan \theta = \frac{F}{R},$$

and when $\frac{F}{R}$ has its greatest value, the angle θ is greatest.

This greatest value of θ is called the **Angle of Friction**, and is generally denoted by λ .

In other words, the angle of friction is the maximum angle that the resultant reaction can make with the normal.

Also we have $\tan \lambda = \frac{F_{\max}}{R} = \mu,$

giving the relation between λ and μ .

If a body is on a plane and the plane is tilted up, the body will be on the point of sliding, or will begin to slide when the inclination of the plane is equal to the angle of friction; for the resultant reaction must balance the weight, and the forces are as in Fig. 29, and by geometry

$$\lambda = \alpha.$$

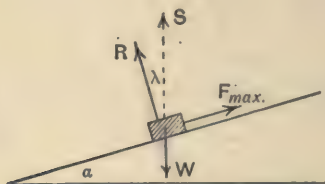


FIG. 29.

Hence the angle of friction between a body and a movable plane can be readily found in this way, and the coefficient of friction deduced by the relation

$$\mu = \tan \lambda.$$

20. Smooth and Rough Pegs.

When a string passes round a fixed peg and there is no friction between the peg and the string, the peg is said to be smooth, and the tension of the string is the same throughout, but if the peg is not smooth the tension will not necessarily be the same on both sides of the peg. For example, if two weights, each W , hang by a string over a peg, the addition of a small weight w on one side may not cause the weights to move, and in this case the tension of the string on one side will be W , and on the other $W + w$.

The term *Smooth Pulley*, which is often used (but rather incorrectly), must be understood in Statics in the same sense

as a smooth peg, namely a body which will let a string pass round it without any friction being exerted, in which case, however, the pulley would not be turned round by a pull in the string.

21. Smooth Joints.

If a bar is hinged at one end to a fixed point, and the joint is spoken of as frictionless or smooth, it is equivalent to saying that the pressure or tension exerted through the hinge passes through the fixed point. For the joint may be considered as formed by a pin running through the bar, the hole in the bar being slightly larger than the pin. This, the usual kind of joint, is called a *Pin Joint*.

Suppose the bar touches the pin at one point A only; then,

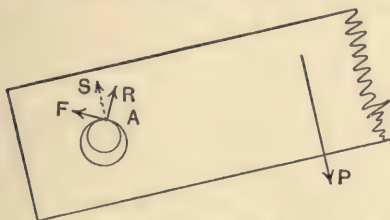


FIG. 30.—Rough joint.

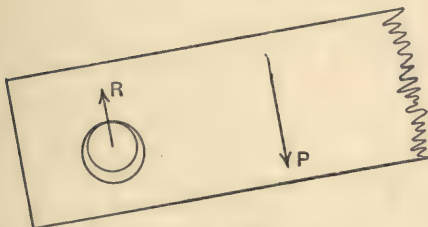


FIG. 31.—Smooth joint.

if a force P tends to turn the bar round, the normal reaction R and the friction F will act at A as shewn. The resultant

reaction will not therefore act through the centre of the pin, but if there were no friction it would do so.

The difference between the cases when there is friction at a joint and when there is not, will be seen better after dealing with moments of forces in Chapter V.

22. Methods of attacking Problems.

Summing up the work of the chapter, we find that when there are only coplanar concurrent forces acting, there are two ways of dealing with cases where there are only three forces, namely by

- (i) the triangle of forces, or its alternative form, Lami's theorem,
- (ii) resolving in two directions.

For more than three forces the method of resolving is usually the only one available, but occasionally two of the forces may be conveniently reduced to one, and thus a question with four forces reduced to one with three.

The work of the chapter may be illustrated by a number of examples.

Examples. 1. A uniform rod AB, weighing 2 lbs., rests with one end A against a vertical wall, and is kept at rest by a string BC attached to a point C in the wall above A. If the rod makes an angle 60° with the wall, find the tension of the string and the reaction at A, supposing there is no friction; and compare the length of the string with that of the rod.

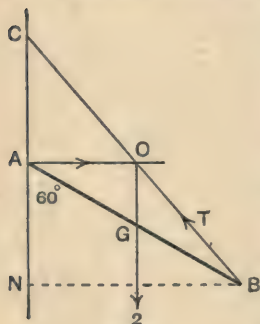


FIG. 32.

There are three forces, the reaction R, the weight 2 lbs., and the tension T. R is horizontal because there is no friction, the weight is vertical, and T must pass through their point of intersection, for, since there are only three forces and two of them meet in a point, the third passes through the same point. Hence, in the figure, if BN is perpendicular to CA, CNB is a triangle of forces, the

weight being parallel to CN, R parallel to NB, and T along BC, and we have by geometry,

since G is the middle point of AB,

$\therefore$ O is the middle point of BC ;

$\therefore$ A is the middle point of CN.

Let

$$AB = 2a ;$$

$$\therefore AN = a,$$

$$CN = 2a,$$

$$NB = a\sqrt{3},$$

$$CB = \sqrt{CN^2 + NB^2} = a\sqrt{7}.$$

Hence

$$\frac{T}{CB} = \frac{R}{NB} = \frac{2}{CN}$$

or

$$\frac{T}{a\sqrt{7}} = \frac{R}{a\sqrt{3}} = \frac{2}{2a} ;$$

$$\therefore T = \sqrt{7} = 2.65 \text{ lbs.-wt.},$$

$$R = \sqrt{3} = 1.73 \text{ lbs.-wt.},$$

$$\frac{CB}{BA} = \frac{\sqrt{7}}{\sqrt{3}} = \sqrt{2.33} = 1.53.$$

2. A uniform rod AB, weighing 4 lbs., is supported by a smooth hinge at A, and is held at an angle of 60° with the downward vertical, by a force acting at B perpendicular to the rod. Find the magnitude of this force and the magnitude and direction of the force at A.

Notice particularly the difference between this example and the last. In Example 1 the end of the rod rested against a smooth vertical wall, and therefore the reaction there was perpendicular to the wall. In the present example the rod is smoothly hinged at A, and therefore all we know at first is that the reaction passes through A in some unknown direction.

With the letters of the figure, the weight and the unknown force P meet in a point O. The third force must therefore pass through O, and consequently acts along OA. If AD is vertical, DOA will be a triangle of forces, and we will first calculate the lengths of its sides,

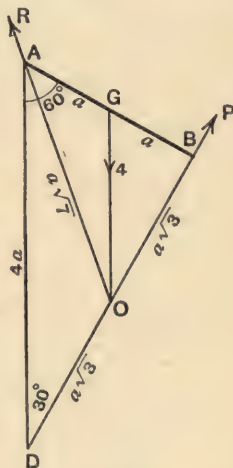


FIG. 33.

Let

$$AB = 2a :$$

$$\therefore DB = 2a\sqrt{3} ;$$

$$\therefore DO = a\sqrt{3} ;$$

also

$$OB = a\sqrt{3} ;$$

$$\therefore OA = a\sqrt{7} ;$$

$$\therefore \frac{P}{a\sqrt{3}} = \frac{R}{a\sqrt{7}} = \frac{4}{4a} ;$$

$$\therefore P = \sqrt{3} = 1.73 \text{ lbs.-wt.},$$

$$R = \sqrt{7} = 2.65 \quad ,$$

also the angle R makes with the rod is given by

$$\tan OAB = \frac{\sqrt{3}}{2} = 0.866 ;$$

$$\therefore OAB = 40^\circ 54'.$$

3. A flexible chain, weighing 40 lbs., hangs in a curve with the two ends at the same level, and the tangents at the ends making an angle 30° with the horizontal. Find the tension at the ends and at the lowest point.

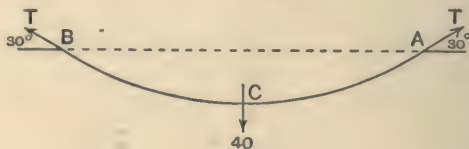


FIG. 34.

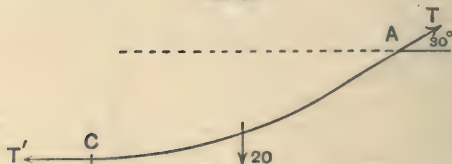


FIG. 35.

The chain hangs symmetrically, and the tensions at the two ends are the same. We can get these tensions by thinking of the chain as a whole, and noticing that we have three forces acting on the chain, namely, the reactions of the supports, each T (the same as the tension there), and the weight of 40 lbs.

Resolving vertically,

$$2T \sin 30^\circ = 40 ;$$

$$\therefore T = 40 \text{ lbs.-wt.}$$

To get the tension at the lowest point we can consider half the chain by itself (Fig. 35). If we notice that the part AC is at rest and acted on by the reaction T , a horizontal pull T' due to the portion CB, and its weight of 20 lbs.; $\therefore$ resolving horizontally,

$$T \cos 30^\circ = T',$$

$$T' = 40 \frac{\sqrt{3}}{2} = 34.64 \text{ lbs.-wt.}$$

The solution may be equally well obtained by Lami's Theorem.

4. A body, of weight 100 lbs., is to be pulled up an incline of 3 (vertical) to 4 (horizontal) by a force parallel to the plane. Find the least force if the coefficient of friction is $\frac{1}{4}$.

There are four forces acting, so we resolve in two directions, the most convenient being parallel and perpendicular to the plane.

Perpendicular to the plane,

$$R = 100 \cos \alpha = 100 \times \frac{4}{5} = 80,$$

$$F = \frac{1}{4}R = 20.$$

Parallel to the plane,

$$P = F + 100 \sin \alpha = 20 + 100 \times \frac{3}{5} = 80 \text{ lbs.-wt.}$$

5. A body of weight W is to be pulled up a plane of inclination α by a force making an angle θ with the plane. If the angle of friction is λ , find the force, and find what θ must be, so that the required force may be the least possible.

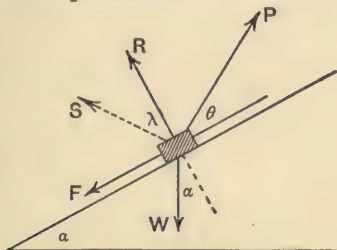


FIG. 37.

We may proceed by resolving as in Ex. 4, but we can get the result very easily by replacing the normal reaction and friction by the resultant reaction S , which, when the body is on the point of moving,

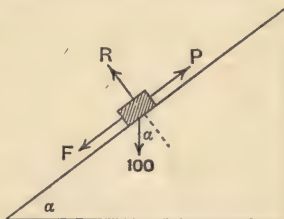


FIG. 36.

acts an angle λ with the normal. We then have three forces only, and can use Lami's Theorem. We have then

$$\frac{P}{\sin(180 - \alpha - \lambda)} = \frac{S}{\sin(90 + \alpha + \theta)} = \frac{W}{\sin(90 + \lambda - \theta)},$$

$$\frac{P}{\sin(\alpha + \lambda)} = \frac{S}{\cos(\alpha + \theta)} = \frac{W}{\cos(\theta - \lambda)}$$

$$P = W \frac{\sin(\alpha + \lambda)}{\cos(\theta - \lambda)}.$$

Hence, for different values of θ , P is least when $\cos(\theta - \lambda)$ is greatest, that is, when $\theta = \lambda$,

or

P is at right angles to S ,

and

$$P_{\min} = W \sin(\alpha + \lambda).$$

23. Particles connected by a String.

When two particles are connected by a string it is generally necessary to deal with each separately, noticing that the string pulls each body with the same force. The student should therefore mark the tension as it acts on each body, and form the equations for each body by Lami's Theorem or otherwise.

6. Two bodies of weights W , W' rest on two smooth inclined planes placed back to back, and of inclinations α and β , being connected by a string passing over a smooth peg at the top of the planes. Find the ratio of the weights.

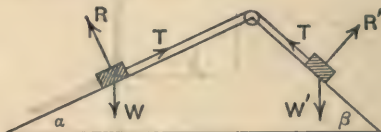


FIG. 38.

The forces are as marked on the figure.

For the weight W , resolving parallel to the plane,

$$T = W \sin \alpha.$$

For W' similarly,

$$T = W' \sin \beta;$$

$$\therefore W \sin \alpha = W' \sin \beta,$$

$$W/W' = \sin \alpha / \sin \beta.$$

7. AB, BC, CD are equal portions of a string, whose ends A, D are fixed, and weights W , W' are hung at B and C, such that AB is horizontal, and BC, CD make angles of 30° and 60° respectively with the horizontal. If $W = 10$ lbs., find W' and the tensions in the three portions of the string.

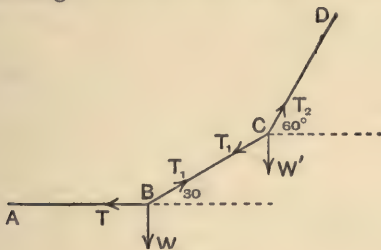


FIG. 39.

If the tensions are T , T_1 , T_2 , as in the figure, then T , T_1 , W balance at B, and T_1 , T_2 , W' balance at C.

Applying Lami's Theorem,

at B,
$$\frac{T}{\sin 120^\circ} = \frac{T_1}{\sin 90^\circ} = \frac{10}{\sin 150^\circ};$$

at C,
$$\frac{T_2}{\sin 60^\circ} = \frac{T_1}{\sin 150^\circ} = \frac{W'}{\sin 150^\circ};$$

whence

$$T_1 = W' = 20,$$

$$T = 10\sqrt{3} = 17.3,$$

$$T_2 = 20\sqrt{3} = 34.6 \text{ lbs.-wt.}$$

Notice that the lengths of the portions of string do not affect the results.

8. A uniform rod rests with its ends on two smooth inclined planes of inclinations α and β . Find the angle it makes with the vertical.

The reactions at the ends of the rod act perpendicularly to the planes, the weight W acting at the middle point of the rod (since it is uniform) is vertical. As there are only these three forces, the figure is to be drawn with their lines of action meeting in a

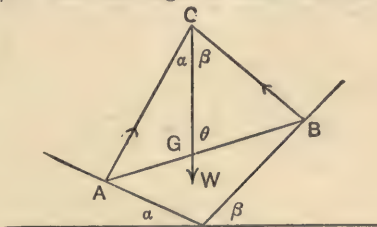


FIG. 40.

point. If θ is the angle the rod makes with the vertical, θ can be found as follows:

From the triangle AGC,

$$\sin \angle GAC / \sin \angle ACG = CG / AG,$$

$$\sin (\theta - \alpha) / \sin \alpha = CG / GA.$$

Similarly, from the triangle BCG,

$$\sin (\theta + \beta) / \sin \beta = CG / GB,$$

$$\sin (\theta - \alpha) / \sin \alpha = \sin (\theta + \beta) / \sin \beta,$$

$$\sin \theta \cot \alpha - \cos \theta = \sin \theta \cot \beta + \cos \theta,$$

$$2 \cos \theta = \sin \theta (\cot \alpha - \cot \beta),$$

or

$$2 \cot \theta = \cot \alpha - \cot \beta.$$

9. OA and OB are two straight lines. Forces acting along them are represented in magnitude by $h \cdot OA$ and $k \cdot OB$. Shew that their resultant is $(h+k)OG$, where G is the point which divides AB in the ratio $k : h$.

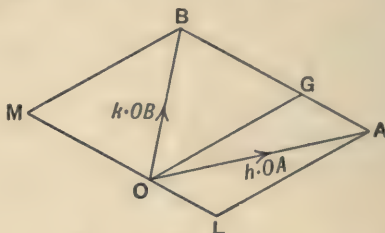


FIG. 41.

Take G so that $AG/GB = h/k$, and draw LOM parallel to AB, making $OL = AG$ and $OM = GB$, so that OGAL and OGBM are parallelograms. Then the force $h \cdot OA$ can be replaced by components

$$h \cdot OL \text{ and } h \cdot OG,$$

and $k \cdot OB$ can be replaced by

$$k \cdot OM \text{ and } k \cdot OG.$$

Now, since

$$LO/OM = k/h,$$

$$h \cdot LO = k \cdot OM,$$

and the forces $h \cdot OL$ and $k \cdot OM$ balance one another, and we are left with a force

$$(h+k)OG.$$

24. Graphical Solutions.

Examples may often be solved to a sufficient degree of accuracy for practical work by graphical methods where the accurate solution is troublesome.

For example, if we wanted an approximate solution of No. 1, Art. 22, we have only to draw the figure on a fairly large scale and measure off the sides of the triangle AOC.

Thus, draw first the vertical AN, and AB at an angle 60° with it. Make AN and AC each = 2 inches; draw NB horizontal, and finish off the figure as in the diagram. Then the sides AO, OC will represent the reaction and tension on the scale of 1 lb.-wt. to the inch and the angle ACB can be measured off by a protractor.

Examples. 1. A weight of 100 lbs. on a plane of inclination 20° is just pulled up the plane by a force P acting at 25° with the plane. Find P if the angle of friction is 35° .

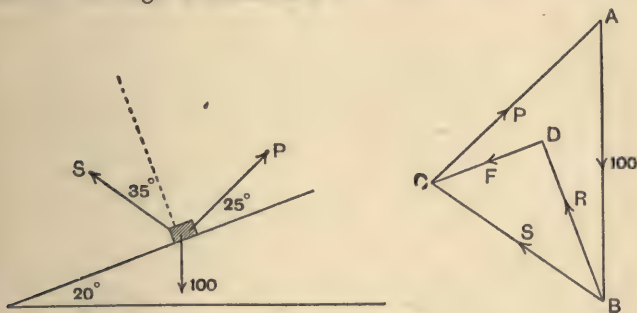


FIG. 42.

The angles between the directions of the three forces acting are known, and the magnitude of one (the weight) is known. Hence we can draw a triangle with one side AB 2.5 ins. long (say) to represent the weight of 100 lbs., and the angles BAC, ABC 45° and 55° . Or draw diagram 42 accurately first and then a triangle with its sides parallel to W, P and S. Then AC will represent P on the same scale of 1 in. to 40 lbs. weight. Also BC will represent the resultant reaction. The normal reaction and friction can be got by drawing BD perpendicular to the plane and CD parallel to it, when BD will represent the normal reaction and DC the friction. Scaling these off, we find $CA = 2.07$, $BD = 1.46$, $DC = 1.03$, $BC = 1.79$; hence

$$\begin{aligned} P &= 82.8. \\ \text{Normal reaction} &= 58.4. \\ \text{Friction} &= 41.2. \\ \text{Resultant reaction} &= 71.6. \end{aligned}$$

2. A rod AB of length 4 ft. and weight 15 lbs. is supported by two strings attached to a point C and of lengths 2.5 and 3.5 ft. respectively. Find the tension of the strings and the angles they make with the vertical.

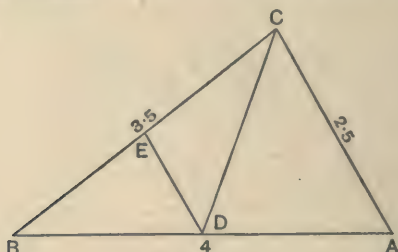


FIG. 43.

Here the sides AB, BC, CA of the triangle are given, and if D is the middle point of AB, CD will represent the vertical, since the weight acting at D must also act through C where the other two forces intersect. If now E is the middle point of BC and DE is drawn, DEC being a triangle of forces, the tensions and weight are in the ratio DE, EC, CD. Scaling off, we find $CD = 2.3$, and as this represents 15 pounds DE and EC represent 8.2 lbs. and 11.4 lbs. Also the angles ACD, BCD, ADC, which the strings and rod make with the vertical, can be read off with the protractor and are 48° , 43° , 71° approximately.

3. A heavy rod rests with its two ends on two smooth planes of inclinations 40° and 70° . Find the angle it makes with the vertical.

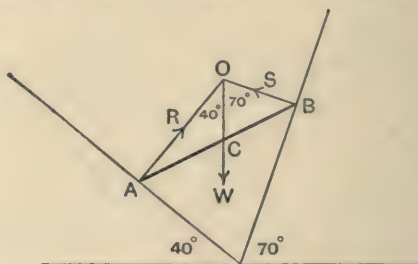


FIG. 44.

The figure has to be drawn so that the reactions at A and B and the weight at C (the middle point of AB) meet in a point. By

geometry the angle $AOC = 40^\circ$, $BOC = 70^\circ$. Hence we must construct the figure by taking any line AB for the rod and on AC describe a segment containing an angle of 40° and on BC a segment containing 70° . If these segments meet in O, OC represents the vertical and the angle OCB can be read off, and is about 67° .

EXAMPLES.

1. A rod AB, of weight 20 lbs., is supported in a horizontal position by a smooth hinge at A, and a string at B, making an angle 45° with the vertical. Find the tension of the string and the magnitude and direction of the reaction at A.

2. A rod AB, of weight 10 lbs., is supported at the end A by a smooth hinge, and is held at an angle of 60° with the downward vertical, by means of a string attached at B, and making an angle 30° with the rod. Find the tension of the string and the reaction at A in direction and magnitude.

3. A rod AB, of weight 5 lbs., is supported by a smooth hinge at A, and rests against a smooth peg at B. If AB makes an angle of 30° with the vertical, find the pressure at B, and the reaction at A in magnitude and direction.

4. Solve Question 3, if the peg, instead of being at B, is three-quarters of the way along from B to A.

5. A rod AB, of weight 10 lbs., is supported by a smooth hinge at A and by a string at B passing over a pulley at C vertically above A and carrying a weight of 5 lbs. at its end. If $AC = AB$, find the angle B makes with the vertical and the direction and magnitude of the reaction at A.

6. A rod, 40 inches long and weighing 8 lbs., rests with one end A against a smooth vertical wall, and to B is attached a string which is tied to a point in the wall 24 inches above A. Find the length and tension of the string, the reaction at A, and the angle the rod makes with the wall.

7. A thin rod, of length $2l$, rests with one end pressing against a smooth vertical wall, being supported by passing over a smooth peg at a distance c from the wall. Shew that it must make an angle with the wall given by $\sin^3 \theta = c/l$.

8. A rod AB, of weight W, is supported with one end A against a smooth vertical wall, and by a string attached to it at C and to a point in the wall. If $AC = \frac{1}{4}AB$, shew that the length of the string is also $\frac{1}{4}AB$, and if the rod makes an angle θ with the wall, find the tension of the string.

9. A sphere, of weight W and radius r , rests on a smooth plane of inclination α , being kept at rest by a horizontal string attached to the sphere and to a point in the plane. Find the length and tension of the string, and the reaction at the point of contact.

10. A sphere, of weight 3 lbs., rests in the angle between two planes, one of which is vertical and the other inclined at an angle of 30° to the vertical. Find the pressures on the planes.

11. A sphere, of weight W , rests on two planes of inclinations α , β . Find the pressures on the planes.

12. Two spheres, of weights W , W' , rest one on each of two planes placed back to back of inclinations α , α' , being connected by a string which runs horizontally from one to the other. Shew that

$$W \tan \alpha = W' \tan \alpha'.$$

13. A rod, of weight 6 lbs., rests with its ends on two smooth planes of inclinations 30° and 60° . Find the angle the rod makes with the vertical, and find the pressures on the planes.

14. A rod rests with its ends on two smooth planes of inclinations 40° and 70° . Find the angle that it makes with the vertical.

15. ABCD is a string, and equal weights W are attached at B and C, and BC remains horizontal, while AB, CD make angles of 60° with the horizontal. Find the tensions.

16. A small ring, of weight W , can slide on a smooth vertical circle, and is attached to the highest point by a string which subtends an angle 2α at the centre. Find the tension of the string.

17. What is the least force parallel to the incline which will pull a body of weight 100 lbs. up an incline of 30° if the coefficient of friction is 0.2?

Find also the least force that will prevent it from sliding down.

18. A plane slopes at 5 (vertical) to 12 (horizontal). Find the greatest and least forces parallel to the plane which can act on a body of weight 26 lbs. on the plane without its moving, if the coefficient of friction is 0.25.

19. Find the least force (in any direction) necessary to move a body of weight W along a horizontal plane if the coefficient of friction is 0.75.

20. Find the least force required to move a body of weight 20 lbs. along a horizontal plane if the coefficient of friction is 1.0.

21. Find the least horizontal force required to move a body of weight 50 lbs. up an incline of 30° if the coefficient of friction is 0.5.

22. Find the least horizontal force required to move a body of weight 120 lbs. up an incline of 25° if the coefficient of friction is 0.3.

23. Shew that the least force necessary to stop a body from running down a plane of inclination α , when the angle of friction is λ , must be applied at an angle λ with the plane ($\lambda < \alpha$).

24. What force, applied at an angle of 15° to the plane, will be required to drag a body of weight 140 lbs. up an incline of 35° if the coefficient of friction is 0.5 ?

25. A wheel, of radius r and weight W , is to be dragged over a rectangular block of stone of height h ($< r$), by means of a horizontal force applied at its centre. Find the magnitude of the force when the lowest point of the wheel is at a height k above the ground, and find the smallest force P required.

Examine the case when $r = 30$ inches, $h = 18$ inches, $W = 50$ lbs.

26. Equal particles are hung from a series of points on a weightless string. Shew that if $\theta_1, \theta_2, \dots$ are the angles the successive portions make with the horizontal, $\tan \theta_1, \tan \theta_2, \dots$ are in A.P., and that the horizontal components of the tensions are the same.

27. If in Question 26 the points of attachment of the weights are such that the projections on the horizontal of the separate portions of the string are each equal to a , and the first portion starting from the lower end is horizontal with a tension T_0 in it, and the suspended weights are each W , prove that the x and y coordinates of the n^{th} angle measured from the lower end are na and $\frac{n(n-1)Wa}{2T_0}$, and hence that the angles lie on a parabola.

28. A block, of weight W , rests on a rough horizontal plane. A piston presses down upon it with force P . If the coefficient of friction is μ at each contact, what is the least horizontal force required to draw out the block ?

29. AB is a weightless rod carrying a weight W at a point which divides the rod in the ratio $a : b$. It is supported by strings at the ends, making angles α and β with the vertical. Shew that the rod makes an angle θ with the vertical given by

$$(a+b) \cot \theta = b \cot \beta - a \cot \alpha,$$

and find the tensions of the strings.

30. A weight w rests on a rough horizontal table, the coefficient of friction being μ . To it are attached two springs which pass over pulleys and carry weights W at the end of each. The strings each make an angle α with the horizontal. If an extra weight W is added to one of these, shew that in limiting equilibrium

$$W = \mu(w - 2W \sin \alpha) / (\cos \alpha + \mu \sin \alpha).$$

31. A ring, of weight w , slides on a smooth vertical circular wire of radius a , being attached to the extremity of the horizontal diameter by a string of length $a/2$. Shew that the tension is approximately $0.9w$, and find the pressure.

32. A weight of 100 lbs. is supported on a rough plane of inclination 30° , by a rope at an inclination 45° with the horizontal. If the coefficient of friction is 0.25, find the greatest and least tensions of the rope consistent with equilibrium.

33. ABC is a triangle, and D is a point in BC. Forces P, Q along AB, AC have a resultant R along AD. Prove that

$$P/R = CD \cdot AB/BC \cdot AD.$$

34. A ring C slides on a string whose ends are 2 ft. apart, AB making an angle 30° with the horizontal. Shew that in the position of equilibrium the two portions of the string make equal angles with the vertical, and if the length of the string is $2\sqrt{3}$ ft., shew that these angles are 30° .

Solve the following questions graphically :

35. A rod AB, of weight 20 lbs., is smoothly hinged at A, and is held at an angle 50° with the upward vertical by a string attached to B and to a point above A and making an angle 20° with the vertical. Find the tension of the string and the reaction at A.

36. A rod AB, of length 2 ft. and weight 10 lbs., is supported by a smooth hinge at A and by a string attached at B which passes over a smooth pulley at C vertically above A, and supports a weight of 5 lbs. at its end. If AC is 4 ft., find the angle the rod makes with the vertical and the magnitude and direction of the reaction at A.

37. AB is a smooth rod fixed at an angle 40° with the horizontal. A small ring of weight 15 oz. can slide on AB and is attached by a string 3.5 feet long to a point C vertically above A. If CA = 4 ft. find the tension of the string and pressure of the rod.

38. A sphere, of weight 15 lbs., is kept at rest on a plane of inclination 18° by a horizontal string attached to its highest point. Find the tension of the string.

39. A rod, of weight 3 lbs. and length 2.3 ft., is hinged at one end to the lowest point of a fixed vertical circle of radius 2 ft., and the other end rests on the inside of the circle. Find the reactions at each end.

40. A weight C, of 100 lbs., is hung by two wires attached to points A and B 6 ft. apart, B being 2 ft. above A. If the lengths of the wires are 4 and 5 ft. respectively, find their tensions.

41. A ring C, of weight 2 lbs., can slide on a string 5 ft. long, whose ends A and B are 4 ft. apart, and the end B is 1 ft. above A. Find the tension of the string and the lengths of AC, CB in the position of equilibrium.

CHAPTER III.

PARALLEL FORCES.

25. We have seen how to deal with forces acting at one point, basing the work on the parallelogram of forces. If two forces are parallel we cannot directly apply the parallelogram law, but we will see that this law is sufficient, with the principle of the transmissibility of force, to obtain the resultant in this case also.

Thus, suppose two parallel forces P and Q act at points A , B of a body. The effect on the body is not altered by

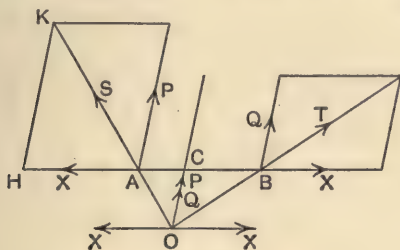


FIG. 45.

introducing two forces, each X , at A and B , in opposite directions along AB . The forces at A can now be compounded into a single force S , and those at B into a single force T , so that the original forces P and Q are equivalent in their action to the forces S and T .

S and T , if produced, will meet in general at a point O

(there is an exceptional case to be noted later), and consequently can be compounded into a single force. This is, however, most easily done by first resolving S and T at O into components in their original directions; thus S is replaced by P and X parallel to the original P and X , and T by Q and X . The two X 's now cancel again, and leave a single force $P+Q$ parallel to the original P and Q .

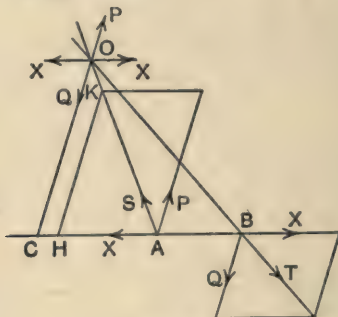


FIG. 46.

It is necessary to find the position of the resultant, and this is best given by finding the point C where it cuts the line AB .

In the figure the triangle AHK is similar to ACO .

$$AC/CO = AH/HK = X/P;$$

similarly

$$BC/CO = X/Q;$$

$\therefore$ dividing,

$$AC/BC = Q/P. \dots\dots\dots(1)$$

To help to follow the argument we will represent the steps by different figures.

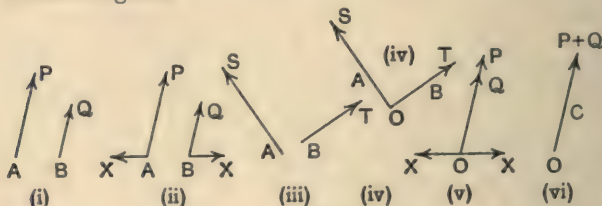


FIG. 47.

The beginner should follow these figures, and notice carefully that the forces in the first are exactly equivalent to those in the next, and so on.

26. If a force R balances two parallel forces P and Q , it must consequently be equal to $P+Q$, and act through C , and since by (1)

$$\frac{P}{BC} = \frac{Q}{AC},$$

$$\text{each fraction} = \frac{P+Q}{BC+CA} = \frac{R}{AB}$$

or

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB} \dots\dots\dots (2)$$

In these results it does not matter where the points A and B are on the lines of action of the forces, and in most cases it is convenient to take AB perpendicular to the forces.

We may express (2) in words thus :

If three parallel forces are in equilibrium, each is proportional to the distance between the other two.

This is analogous to Lami's Theorem for forces acting at one point.

27. If $AB = a$

and $AC = x$,

then, since $P \cdot AC = Q \cdot CB$,

$$Px = Q(a-x),$$

$$x = \frac{Qa}{P+Q} \dots\dots\dots (3)$$

All the above results are true if P and Q are in opposite directions, provided they are not numerically equal, as will be seen from Fig. 46, and the equation (3), giving the position of the resultant, still holds if proper signs are given to P and Q .

Example. Find the magnitude, position and direction of the resultant in the following cases :

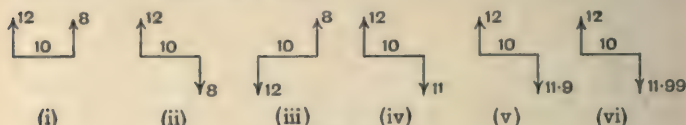


FIG. 48.

If parallel forces P and $P+p$ act in opposite directions at a distance a apart, (3) gives

$$x = \frac{-(P+p)a}{P-(P+p)} = \frac{P+p}{p} a = a + \frac{P}{p} a,$$

and the smaller p is, the larger is x ; or the more nearly equal the forces are, the further off, and at the same time the smaller is the resultant.

If $p=0$, the forces are each P , and the forces cannot be reduced to a single resultant. This will also be seen from the

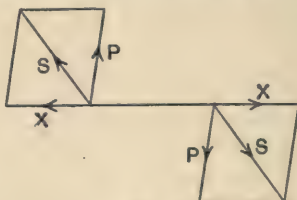


FIG. 49.

fact that the method of finding the resultant in this case fails, for if P and Q are equal, parallel and opposite, S and T are also equal, parallel and opposite, and no progress has been made by the introduction of the forces X .

28. Couples.

Two equal, parallel and opposite forces, being incapable of being replaced by a single resultant, are said to form a couple.

The properties of couples will be discussed in the next chapter.

Examples. 1. Find the magnitude and position of the resultant of the following pairs of parallel forces in the same direction :

- (i) 10 lbs. and 6 lbs., 8 inches apart.
- (ii) 8 lbs. and 4 lbs., 20 inches apart.
- (iii) 5 lbs. and 4 lbs., 100 inches apart.

2. Find the resultants of the pairs of forces as in Question (1), but with the forces acting in opposite directions.

3. A weightless rod, of length 15 inches, carries a weight of 10 lbs., 4 inches from one end, and is supported in a horizontal position by spring balances attached to the ends of the rod. What will the readings of the spring balances be ?

4. A beam AB, 16 feet long and weighing 120 lbs., rests on two supports, one 1 foot from A, and the other 3 feet from B. Find the pressure on each support.

5. A rod AB, of length 6 feet and weight 15 lbs., can turn in a vertical plane about a smooth pin 30 inches from the end A. The rod is kept at any inclination to the vertical by a vertical string passing downwards from A. Find the tension of this string and the reaction at A.

6. If, in Question 5, the string can bear a maximum tension of 20 lbs.-wt., what is the greatest extra weight that can be hung at the centre of the rod without breaking the string ?

7. A rod AB, 5 feet long and of weight 20 lbs., is supported by a smooth hinge at A, and by a vertical string at C, where AC is 18 inches. Find the tension of the string.

If the maximum tension the string can bear is 40 lbs., find the greatest extra weight that could be hung at the centre of the rod without breaking the string.

8. A man carries a weight of 25 lbs. at one end of a stick 3 ft. 6 in. long, resting on his shoulder, and holds the other end. Neglecting the weight of the stick, find the pressure on his shoulder when the length of the stick projecting in front of his shoulder is (i) 2 ft., (ii) 2 ft. 6 ins.

29. Moments.

In trying to obtain the resultant of a number of parallel forces, it is useful to introduce the idea of Moment of a force.

Still dealing with forces in one plane, we can define the *Moment of a Force about a Point* in the plane, as *the product of the force into the perpendicular distance of the point from the line of action of the force.*

Thus, in the figure the moment of P about X is $P \cdot XN$.

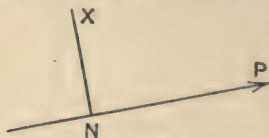


FIG. 50.

If the force is in lbs.-wt. and the distance in feet, the moment will be said to be in lb.-wt.-ft. units. Thus, if P is 10 lbs.-wt. and XN 6 ins., the moment is 5 lb.-wt.-ft. units, or simply 5 lb.-wt.-ft.

The moment of a force is a measure of its tendency to turn a body about the point. Thus, if a rod was hinged at X and the force P acts on it, the tendency to make it turn will be proportional both to the magnitude of P and to the distance XN .

Note that a force has no moment about a point in its line of action, and conversely, if the moment of a force about a given point is zero, the force must pass through the point or be itself zero. Also, if the moment about X is not zero, the force cannot pass through X .

Moments may be positive or negative, a positive moment being the moment of a force which tends to make the body

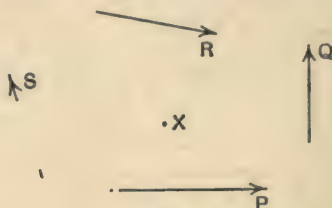


FIG. 51.

rotate in a positive direction about the point. As usual, either direction may be considered positive, and the opposite one is then negative. To decide whether the moment of a given force is to be regarded as positive or negative, it is easiest

to consider in which direction the force would turn a body about a hinge placed at the given point. Thus, in Fig. 51, if the moment of P about X is taken as positive, then the moment of Q is also positive, while the moments of R and S are negative.

30. If we take any point A on the line of action of P (Fig. 52), and join AX , the moment of P about X is

$$P \cdot XN = P \cdot AX \sin \theta = P \sin \theta \cdot AX.$$

Now $P \sin \theta$ is the resolute of P perpendicular to AX . If, therefore, we replace P by its resolutes along and perpendicular

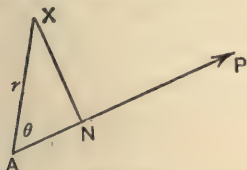


FIG. 52.

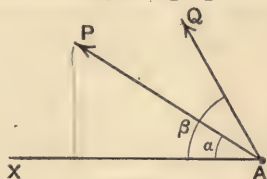


FIG. 53.

to AX , the resolute along AX has no moment about X , while the resolute perpendicular to AX has the same moment as P itself. This may be regarded as a special case of the **Principle of Moments**, which may be stated in general in the following way:

If two forces have a resultant, the moment of the resultant about any point is equal to the sum of the moments of the separate forces about the same point.

Case I. For two forces meeting at a point.

The sum of the moments of P and Q about X in Fig. 53

$$\begin{aligned} &= P \cdot AX \sin \alpha + Q \cdot AX \sin \beta \\ &= AX \cdot (P \sin \alpha + Q \sin \beta) \\ &= AX \times \text{sum of the resolutes perpendicular to } AX \\ &= AX \times \text{resolute of resultant perpendicular to } AX \\ &= \text{moment of the resultant about } X. \end{aligned}$$

By Chapter II. this result evidently applies at once to any number of forces.

32. Composition of any Number of Parallel Forces.

To get the resultant of any number of parallel forces, it is only necessary to compound them two at a time. If at any stage we have two equal and opposite forces, we can leave one of them for a time and combine the other with another force and so on, so that finally we can reduce the whole system to a single force or to a couple. But the following method, by the Principle of Moments, is much easier.

33. Algebraic Treatment of Parallel Forces.

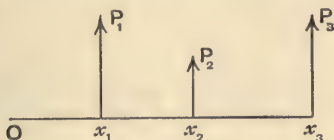


FIG. 56.

Let the forces be P_1, P_2, P_3 , etc., acting at distances x_1, x_2, x_3 , etc., from O. Then, by Art. 31, the resultant will be

$$P_1 + P_2 + P_3 + \dots \text{ or } \Sigma P,$$

and the moment about O will be

$$P_1x_1 + P_2x_2 + P_3x_3 + \dots \text{ or } \Sigma Px.$$

This must be the same as the moment of the resultant. Hence, if the resultant is R and acts at a distance $\bar{x}$ from O,

$$\left. \begin{aligned} R &= \Sigma P, \\ R\bar{x} &= \Sigma Px; \end{aligned} \right\} \dots\dots\dots (4)$$

$$\therefore \bar{x} = \frac{\Sigma Px}{\Sigma P}.$$

For the forces to be in equilibrium,

$$R = 0;$$

$$\therefore \Sigma P = 0$$

and

$$\Sigma Px = 0.$$

Special cases :

(i) If $\Sigma Px = 0$, but ΣP is not $= 0$, we have $\bar{x} = 0$, or the resultant passes through O.

(ii) If $\Sigma P = 0$, but ΣPx is not $= 0$, equations (4) cannot be satisfied ; the forces, in fact, cannot be reduced to a single force, but reduce to a couple. For if $\Sigma P = 0$, suppose we reduce the forces in the positive direction to a single force and those in the negative direction to a single force ; these single forces will be equal. But they cannot be in the same straight line, for then the forces would be in equilibrium and the total moment zero. Hence they must act along different straight lines, and therefore form a couple.

Examples. 1. Prove that if the sum of the moments about two points A and B are separately zero, and AB is not parallel to the forces, the forces must be in equilibrium.

2. Prove algebraically that if $\Sigma P = 0$, and the sum of the moments about one point is zero, the sum of the moments about any other point is zero.

34. Centre of Mass.

If a system of particles in one straight line, of weights W_1, W_2, W_3 , etc., are at distances x_1, x_2, x_3 , etc., from a point O in the line, then by Art. 33 the resultant weight acts at a point G at a distance $\bar{x}$ from O given by

$$\bar{x} = \frac{\Sigma Wx}{\Sigma W}.$$

This point is called the Centre of Mass (or Centre of Gravity) of the weights. Thus, if the weights are fixed on a rigid weightless wire, the whole could be supported by a pressure of ΣW acting upwards at G.

In the same way, if a thin (that is to say indefinitely thin) rod is divided into a large number of little bits, each little bit may be regarded as acted on by its own weight, and these forces will have a single resultant passing through the Centre

of Mass of the rod. If the rod is uniform (that is to say the mass per unit length is the same everywhere), its centre of mass will be at the middle point of the rod, but if the rod is not uniform the centre of mass may be nearer one end. If we are told that the centre of mass of a rod is at a certain point G, it means that the weight of the rod may be considered as a single force acting at G.

It will be noticed from the definition and the properties of parallel forces that the position in which the rod is placed does not affect the position of the centre of mass of the rod.

Other cases of centres of mass will be dealt with later (see Chapter X.).

Examples. 1. A plank AB, 10 ft. long and weighing 50 lbs., rests on two supports, one at A and the other 2 ft. from B. If a man of weight 150 lbs. stands on the plank 3 ft. from A, find the pressure on the supports.

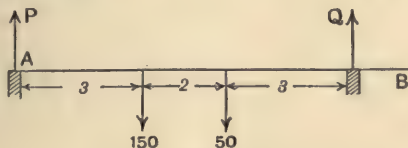


FIG. 57.

Taking the man and plank together, the forces acting on them are the weights and the reactions P and Q of the supports.

$$\therefore P + Q - 150 - 50 = 0,$$

taking moments about A,

$$Q \times 8 - 150 \times 3 - 50 \times 5 = 0;$$

$$\therefore P + Q = 200$$

and

$$8Q = 450 + 250 = 700;$$

$$\therefore Q = 87.5;$$

$$\therefore P = 112.5.$$

2. A uniform rod, 5 ft. long and weighing 6 lbs., has weights 1, 2, 3, 4, 5 attached to it at distances 1, 2, 3, 4, 5 ft. from one end A. Find at what point it can be supported by a single force.

Here

$$R = 1 + 2 + 3 + 4 + 5 + 6 = 21,$$

B.E.S.

D

and taking moments about A,

$$R\bar{x} = 1 \times 1 + 2 \times 2 + 3 \times 3 + 4 \times 4 + 5 \times 5 + 6 \times 2\frac{1}{2} = 71,$$

$$\bar{x} = 71/21 = 3.38,$$

or 3.38 ft. from A.

3. Find the resultant in magnitude and position of the following forces :

(i) 2, -3, -3, 6, -1 lbs.-wt. at intervals of 1 ft. along a rod.

(ii) 2, -3, -5, 6, -1 as before.

(iii) 2, -3, -7, 6, -1 as before.

4. At points along a rod 4 ft. long, forces 2, -3, -4, 6, -1, act at intervals of 1 ft. Prove that the forces are equivalent to a couple, and find its moment.

5. A horizontal rod, 18 ins. long and weighing $1\frac{1}{2}$ lbs., has a weight of 2 lbs. hung at a point 12 ins. from one end. If the ends are supported by spring balances, find the reading of each.

6. A plank, 20 ft. long and weighing 60 lbs., rests on two supports, one at A and the other 3 ft. from B. If a weight of 40 lbs. is placed on the plank 6 ft. from A, find the pressure on each support.

7. A uniform horizontal beam, 20 ft. long and weighing 2 cwt., is supported at its ends, and carries loads of 3, 2, 5, 4 cwt. at distances 3, 7, 12, 15 ft. from one end. Find the pressures on the supports.

8. A rod AB, 10 ft. long and weighing 6 lbs., is supported at A and at another point. A load of 1 lb. is suspended at B, and 5 lbs. and 2 lbs. at points 2 ft. and 6 ft. from B. If the pressure on the support at A is 4 lbs., where is the other support ?

9. Find the magnitude and line of action of the resultant of parallel forces 3, 5, 8 in one direction, and 7 in the opposite, acting at points A, B, C, D in a straight line, where $AB = 1$ ft., $BC = 3$ ft., $CD = 4$ ft.

10. A plank, 15 ft. long and weighing 30 lbs., rests on supports, one at one end A, and the other $2\frac{1}{2}$ ft. from B. A boy walks along the plank, which just begins to tilt when he is a foot from B. Find the weight of the boy.

11. A uniform bar AB, 4 ft. long, is free to turn in a vertical plane about a hinge at A. It is supported at any angle with the horizontal by a vertical string attached to a point of the rod 18 inches from A. Find the greatest weight that can be hung from B without breaking the string, supposing the string cannot support a tension of more than 16 lbs.-wt.

12. A ladder, 40 ft. long and weighing $1\frac{1}{2}$ cwt., has its centre of mass 16 ft. from one end. It is being carried by two men, one of whom supports it at the heavy end, while the other supports it 4 ft. from the light end. How much of its weight does each support ?

35. The Balance.

In the balance and other instruments for weighing we have important examples of bodies acted on by several parallel forces, and we will therefore deal here with some of the principal forms of the instruments.

The balance that is found in physical and chemical laboratories is a modification for exact purposes of old forms. The essential parts may be represented diagrammatically, thus :

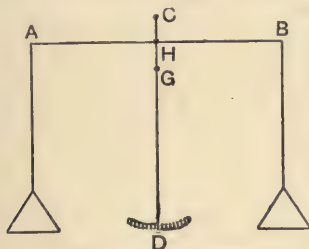


FIG. 58.—The balance.

AB represents the beam of the balance, A and B themselves being the points from which the scale pans hang freely. The beam is constructed so that its centre of mass G is below the line AB. The beam is supported so as to be free to turn about a point C. A pointer rigidly attached to the beam and at right angles to AB moves over a fixed scale D, to shew whether the beam is horizontal. The suspension at C is formed of a knife-edge (or wedge) of steel or agate, fixed to the beam and pointing downwards, and resting on a fixed plate of steel or agate on the pillar of the balance. At A and B similar knife-edges, fixed to the beam, project upwards, and the scale-pans hang from planes which rest on these knife-edges.

By a mechanical release or arrestment below, when a weighing operation is finished the beam can be raised so that the knife-edge at C no longer rests on its plane, and at the same time the planes at A and B are raised off their knife-edges,

and supports below the scale-pans move up for the scale-pans to rest on.

The scale may be a millimetre scale, usually running from 0 to 200, with the 100 at the centre, that is, where the pointer is when the beam is horizontal.

We will suppose at first that the balance is true. This means that the arms AH , HB are equal, the weights of the scale-pans are equal, and C , H , G are in a straight line perpendicular to AB .

In this case, if equal weights P , Q are placed in the scale-pans, the beam will remain horizontal, and the pointer will come to rest at the middle of the scale (*i.e.* at the 100).

However, in accurate weighing it is not usually possible to continue trying different weights in one pan till an exact balance is obtained. Nor is it possible to wait for the balance to come absolutely to rest.

On account of the positions of C and G we shall see that with any given weights in the scale-pans the balance would come to a perfectly definite position of rest, but it actually oscillates about this position, and the position of rest has to be determined from the extreme positions of the pointer in the successive oscillations.

36. Theory of the Balance.

We can now examine the relation between the inclination of the beam and the weights in the pans.

Suppose then that the balance is true, and let

S = the weight of each scale-pan.

P , Q = the weights placed in the pans.

W = the weight of the beam.

a = the length of each arm (that is, $AH = BH = a$).

k = the length CH .

l = the length HG .

θ = the angle AB makes with the horizontal = GCN .

The beam is acted on by the forces

$P + S$, $Q + S$, vertically downwards at A and B,

W , vertically downwards at G,

and the vertical upward reaction at C.

Using the principle that the sum of the moments about any point is zero, take moments about C.

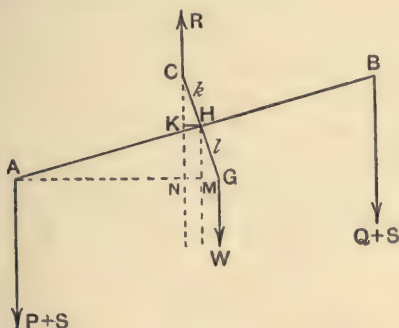


FIG. 59.

In the figure the moment of $P + S$ is $(P + S) \cdot AN$.

Now

$$\begin{aligned} AN &= AM - HK \\ &= a \cos \theta - k \sin \theta; \end{aligned}$$

$\therefore$ moment of $P + S$ is

$$(P + S)(a \cos \theta - k \sin \theta).$$

Also the moment of $Q + S$ is

$$(Q + S)(a \cos \theta + k \sin \theta).$$

The moment of W is

$$W(k + l) \sin \theta.$$

Attending to signs, we have therefore

$$\begin{aligned} (P + S)(a \cos \theta - k \sin \theta) \\ = W(k + l) \sin \theta + (Q + S)(a \cos \theta + k \sin \theta); \end{aligned}$$

$$\begin{aligned} \therefore \sin \theta \{ (P + S)k + (Q + S)k + W(k + l) \} \\ = \cos \theta \{ (P + S)a - (Q + S)a \}; \end{aligned}$$

$$\therefore \tan \theta = \frac{(P - Q)a}{(P + Q + 2S)k + W(k + l)}.$$

From this result it will be seen that if the weights in the scale-pans are equal, the beam remains horizontal, but that if the weights are slightly different, the beam will take a definite position of equilibrium which is not horizontal.

If k and l were both zero, that is, if the centre of mass of the beam and the centre of suspension coincided and were in the line AB, the beam could take any position when equal weights were in the scale-pans, and would turn to a vertical position if the weights were different. This is avoided by having G below AB.

37. Sensitiveness of the Balance.

The ratio $\tan \theta / (P - Q)$ may be taken as a measure of the sensitiveness of the balance, which is therefore

$$\frac{a}{(P + Q + 2S)k + W(k + l)} \dots\dots\dots (5)$$

θ will always be small in any case of weighing, and $\tan \theta$ therefore will be nearly the same as the circular measure of θ , and therefore will also be proportional to the number of divisions on the scale moved over by the pointer from its standard position. Thus the sensitiveness may be denoted by the number of divisions in the displacement of the resting point for a difference of weights of 1 milligramme.

The expression shews that the sensitiveness diminishes as the weights increase.

If however $k = 0$, that is to say, if the three knife-edges are in a straight line, the sensitiveness becomes

$$\frac{a}{Wl},$$

and is independent of the weights in the pans. This condition is generally aimed at, but on account of slight bending of the beam with larger loads it cannot be attained exactly. Usually matters are so arranged that when the pans are unloaded, C

is slightly below AB, while when the maximum load allowed is on the pans, C is above AB. The sensitiveness will then vary slightly with different loads.

In order to make the sensitiveness of a balance as large as possible, it will be seen from the expression (5) that it is necessary to make the arms long, the distances k and l small, and the weights of pans and beam small. These conditions, however, make the time of oscillation long, and for quick weighing some of them have to be given up.

If a balance is not true, if, that is, the arms are not the same length, or the scale-pans not the same weight, we will get a slightly different result for $\tan \theta$, but we will only deal here with the case where the beam is horizontal when unloaded, and in weighing, the beam is brought to horizontality again, or rather the weight is found which would bring it to horizontality.

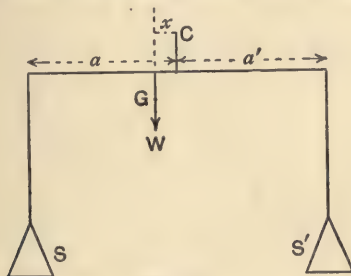


FIG. 60.

38. Double Weighing.

Let the lengths of the arms be a, a' , and the weights of the pans S, S' . Suppose that the balance is horizontal when unloaded, and that the weight of the beam acts at a distance x from C .

Then
$$Sa + Wx = S'a' \dots\dots\dots(1)$$

Now, suppose a weight P in S requires Q in S' to make the beam horizontal,

$$(S + P)a + Wx = (S' + Q)a' \dots\dots\dots(2)$$

Now place P in S', and balance it with a weight Q' in S,

$$(S + Q')a + Wx = (S' + P)a'. \dots\dots\dots(3)$$

From (1) and (2), $Pa = Qa'.$

From (1) and (3), $Q'a = Pa'.$

From these equations we can get both the true weight of P and the ratio of the arms for

$$P^2 = QQ' \dots\dots\dots(4)$$

and $\frac{a'}{a} = \frac{P}{Q} = \sqrt{\frac{Q'}{Q}}. \dots\dots\dots(5)$

In any ordinary case Q and Q' are very nearly equal, and it is sufficiently accurate to take the arithmetic mean for the true weight instead of the geometric mean.

For let $Q' = Q + w,$

where w is very small compared with Q,

$$\begin{aligned} P &= \sqrt{QQ'} = \sqrt{Q(Q+w)} \\ &= Q\sqrt{1+w/Q} \\ &= Q(1+w/2Q) \text{ very nearly} \\ &= Q + w/2 = \frac{Q + Q'}{2}; \dots\dots\dots(6) \end{aligned}$$

also $a'/a = \sqrt{Q'/Q} = \sqrt{1+w/Q} = 1 + \frac{w}{2Q} \text{ very nearly}$

$$= 1 + \frac{Q' - Q}{2Q} = \frac{Q + Q'}{2Q}. \dots\dots\dots(7)$$

39. Steelyards.

The Common or Roman Steelyard consists of a heavy rod AB, supported at a fixed fulcrum C, and having a scale-pan or hook attached at one end to carry the body, whose weight W is required. The rod is graduated, and a weight w can be moved along the rod, and its position when the rod is horizontal gives the weight of the body.

Let S = weight of the scale-pan,
 a = distance of point of support of scale-pan from C,
 x = distance of w from C,
 R = weight of rod,
 b = distance of centre of mass of rod from C.

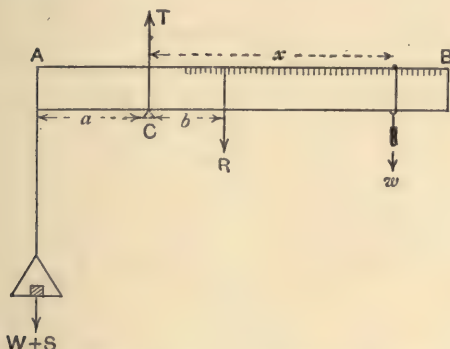


FIG. 61.—The common steelyard.

Then, where there is equilibrium,

$$(S + W)a = wx + Rb. \dots\dots\dots(1)$$

Suppose now that W is altered to W' , and that consequently w has to be shifted to a distance x' to get the rod horizontal again; then we have

$$(S + W')a = wx' + Rb,$$

$$(W' - W)a = w(x' - x).$$

Hence, if we take equal changes in W , say, think of $W' - W$ as equal to a pound for various values of W , then $x' - x$ will always be the same. In other words, the graduations are equally distant all along the bar.

The zero of the graduations will be found by putting $W = 0$. If its distance from C is x_0 , we have

$$x_0 = (Sa - Rb)/w,$$

and hence equation (1) can be written

$$Wa = w(x - x_0)$$

or

$$x - x_0 = Wa/w.$$

If $CB = l$, the greatest weight that can be weighed is

$$w(l - x_0)/a.$$

But if a weight w' can be hung from B, and if we compare two cases, first when w' is not present, and secondly when it is, and suppose w to be in the same position in each case, and if also W_1 is the weight of the body in the second case, we have

$$(S + W)a = wx + Rb,$$

$$(S + W_1)a = wx + Rb + w'l;$$

$$\therefore (W_1 - W)a = w'l;$$

$$\therefore W_1 - W = w'l/a.$$

For example, suppose that in a steelyard

$$AB = 30 \text{ in.},$$

$$AC = 2 \text{ ,,}$$

$$\therefore CB = 28 \text{ ,,}$$

and that the steelyard can weigh anything from zero to 14 lbs., with a moving weight of 1 lb. Now, if we hang a weight of 1 lb. at B, so that

$$w' = 1, \quad l = 28, \quad a = 2,$$

we have

$$W_1 - W = 14,$$

or the steelyard will now weigh anything from 14 to 28 lbs., and a 2-lb. weight hung at B will allow it to weigh anything from 28 lbs. to 42 lbs., and so on.

In the weighing machines for weighing weights to a few hundred-weights this method is adopted, but there are generally additional levers to multiply the effects and to cause the ratio of the weights required at B, to the weight weighed, to be much smaller, so that the weights required for large loads are quite manageable.

A form of weigh-bridge, and also Roberval's Balance, will be found described at the end of the next chapter.

40. The Danish Steelyard.

A much more uncommon form of steelyard is the Danish, where the rod is weighted at one end and is supported by a movable fulcrum, and the weight is given by the position of the fulcrum.

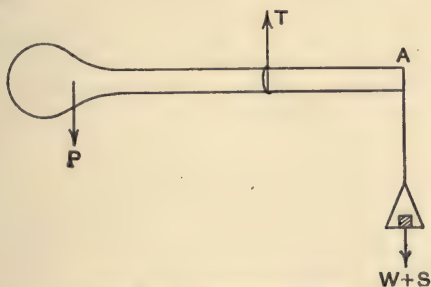


FIG. 62.—The Danish steelyard.

Let P = the weight of the beam,

a = the distance of the centre of mass of the beam from A ,

x = distance of the fulcrum from A when a weight W is in the scale-pan,

S = weight of the scale-pan.

Taking moments about the fulcrum,

$$(S + W)x = P(a - x);$$

$$\therefore Pa = (S + W + P)x.$$

If now W is changed to W' , with a consequent shift of the fulcrum to x' , we have

$$Pa = (S + W' + P)x',$$

$$Pa/x = S + W + P,$$

$$Pa/x' = S + W' + P,$$

$$Pa\left(\frac{1}{x} - \frac{1}{x'}\right) = W - W'.$$

Of course, as W increases, x diminishes, as will be seen from this equation.

If W'' is another weight, and x'' the corresponding value of x ,

$$Pa\left(\frac{1}{x'} - \frac{1}{x''}\right) = W' - W''.$$

Therefore, if

$$W - W' = W' - W'',$$

$$\frac{1}{x} - \frac{1}{x'} = \frac{1}{x'} - \frac{1}{x''},$$

or in other words, if the weights are in arithmetical progression, the distances between the corresponding graduations are in harmonical progression.

Examples. 1. A body appears to weigh 51.474 gms. when placed in one scale-pan of a balance, and to weigh 51.577 gms. when placed in the other. What is its true weight and the ratio of the arms? Also compare the true values given by equations (4) and (5), Art. 38, with the approximate values given by (6) and (7).

2. The three knife-edges of a balance are in one straight line. The length of the balance arm is 20 cms., the mass of the beam is 100 gms. The angular deflection for a change of a centigram in the load is 0.01 of the radian. Find the distance of the centre of mass of the beam from the central knife-edge.

3. A uniform rod AB, of weight w , can turn about a fulcrum at a point whose distances from the ends are a and b . If a body is hung from A it balances a weight W hung from B, and if hung from B it balances a weight W' hung from A. Find the weight of the body in terms of W, W', a, b .

If $a = 12$ ins., $b = 10$ ins., $W = 15$ oz., $W' = 16$ oz., find the weight.

4. The resting point of the pointer of a balance when unloaded is at 96. When a body is being weighed the resting point is at 130 when the weight in the scale-pan is 26.87. When the weight is 26.88 the resting-point is 18. What is the true weight to the nearest milligram?

5. The beam of a balance is 25 cms. long, and the pointer 30 cms. The pointer scale is divided into millimetres. If the weight of the beam is 200 gms. and 1 gm. moves the pointer through one division, shew that the distance of the centre of mass of the beam below the knife-edge is $3/16$ mm., the three knife-edges being in one line.

6. If the arms of a balance are a and a' , and the weights of the scale-pans S and S' , other letters being as in Art. 36, prove that

$$\tan \theta = \frac{Pa - Qa'}{(P + Q + S + S')k + W(k + l)}.$$

7. In a common steelyard the rod AB is a uniform beam of length 36 ins. and weight 3 lbs., and the fulcrum is 4 ins. from A. The scale-pan is weighted so as to be of weight 10 lbs., and the moving weight is 1 lb. It is graduated to quarters of a pound. Find the distance between the graduations, the greatest weight that can be weighed, and the position of the zero.

CHAPTER IV.

WORK. SIMPLE MACHINES.

41. When a man or engine lifts a body vertically, he is said to do work, the amount of work being measured by the product of the weight raised, and the height it is raised. Thus, if a bricklayer takes a hod of bricks weighing 70 lbs. up a vertical ladder 40 ft. high, he is said to do 70×40 ft.-lbs. of work. The work does not depend at all on the time taken to raise the weight. It does not matter whether the man takes 2 mins. or 10 mins. over it, the work done is the same, but the rate of doing the work is different.

More generally, *a force is said to do work when its point of application is moved, and the work done is measured by the product of the force into the distance the point of application moves in the direction of the force.*

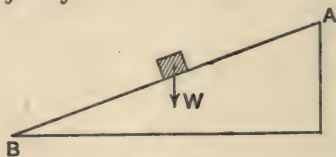


FIG. 63.

Thus, when a body of weight W slides down an incline, gravity acting vertically does work equal to Wh , where h is the vertical height through which it moves, and this is true whether the path taken is straight or curved in any way.

If a force P acts on a particle, and the particle moves a

distance s in a straight line from A to B, while the force remains unchanged in magnitude and direction, always making an angle θ with AB, the work done is $P \cdot s \cos \theta$ or $P \cos \theta \cdot s$, which is equal to the displacement multiplied by the resolute of P in the direction of the displacement.

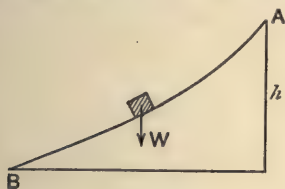


FIG. 64.

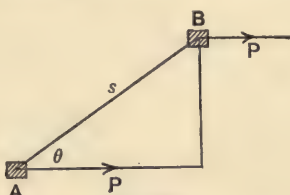


FIG. 65.

If the resolute of the force in the direction of the displacement is negative (in other words, if θ is obtuse), it may be said either that the force does negative work or that work is done against the force.

42. Case of Several Forces.

The work done by the resultant of two forces (constant in magnitude and direction) is equal to the sum of the amounts of work done by the two forces separately.

For, if P and Q are the two forces acting on a particle, and the particle is displaced from A to B, the sum of the amounts of work done by P and Q is

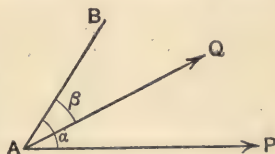


FIG. 66.

$$AB \cdot P \cos \alpha + AB \cdot Q \cos \beta = AB(P \cos \alpha + Q \cos \beta),$$

if α, β are the angles P and Q make with AB , but

$$P \cos \alpha + Q \cos \beta$$

= sum of resolutes of P and Q along AB

= resolute along AB of the resultant of P and Q ;

$\therefore$ the above amount of work is equal to the work done by the resultant. The same result evidently holds for any number of forces meeting at a point.

43. A similar theorem is that the sum of the amounts of work done by a constant force in two successive displacements of the point of application is equal to the work done by the force in the resultant displacement.

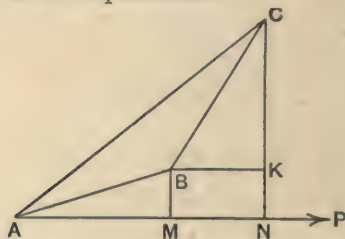


FIG. 67.

For if AB, BC are the two displacements, AC is the resultant displacement, and with the letters of the figure the total work done in the two displacements is

$$\begin{aligned} P \cdot AM + P \cdot BK &= P \cdot (AM + BK) \\ &= P \cdot AN \\ &= \text{work done in the resultant displacement.} \end{aligned}$$

44. Virtual Work.

From Art. 40 it follows that if a particle is in equilibrium under the action of a number of forces, and the particle is displaced in any direction while the forces remain constant in magnitude and direction, the total work done is zero, for the resultant force is zero.

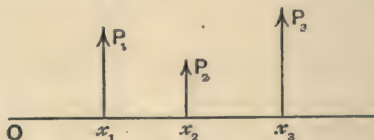


FIG. 68.

The same applies also to a rod acted on by a number of parallel forces in equilibrium, if the rod is displaced either parallel to itself or made to rotate about a point in itself.

For if the forces act at distances $x_1, x_2, x_3, \dots$ from a point O of the rod, we have that

$$\Sigma P = 0$$

and

$$\Sigma Px = 0,$$

but if the rod is displaced parallel to itself through a distance y , the total work done is

$$P_1y + P_2y + \dots = y\Sigma P = 0,$$

and if the rod is rotated through an angle θ about O, while the forces continue to act at the same points of the rod and in the same direction, the work done is

$$P_1x_1 \sin \theta + P_2x_2 \sin \theta + \dots = \sin \theta \Sigma Px = 0.$$

45. In Statics it is often useful to imagine a body at rest as displaced through a small distance, and deduce results by saying that the total work done is zero. Thus, suppose two bodies of weights P and Q connected by a string passing over a frictionless pulley at the top of an incline, one of the bodies (Q) resting on the incline and the other hanging vertically, and suppose there is no friction between Q and the plane, and let the inclination be α .

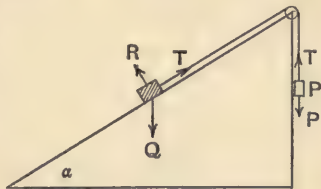


FIG. 69.

We wish to find, by the principle of Virtual Work, the relation between P and Q when they remain at rest. The forces on each body are as marked. Now, suppose P displaced downwards a distance x , Q will move up the same distance. The total work done for each body is zero. The tension does positive work on Q and an equal amount of negative work on P , the work done by R is zero, as the displacement is perpendicular to R ; hence the total work done by gravity on the two bodies is zero, or as it is commonly expressed, the work done

by gravity on P is equal to the work done against gravity by Q . Now Q moves a vertical height of $x \sin \alpha$.

$$\therefore Px = Qx \sin \alpha ;$$

$$\therefore P = Q \sin \alpha.$$

The principle used in this method is called the Principle of Virtual (that is, imagined) Work, and is of frequent application to examples.

We will take one or two examples from what are called simple machines, leaving more advanced examples to a later chapter.

46. A machine is an apparatus in which work is done on the machine at one part (or as often expressed, energy is supplied at this part) with the object of getting work done by the machine at another part.



The force applied to do work at the first part is generally called the Power. The force exerted by the second part of the machine overcomes an external force called the Weight or Resistance. Thus, in the example above, the inclined plane may be regarded as a simple machine in which P is the power and Q is the weight.

Take as another illustration a system of pulleys, represented diagrammatically in Fig. 70, but suppose there are n portions of the string passing between the two blocks. Actually the pulleys are placed beside one another in two blocks, the upper block is spoken of as fixed, the lower, to which the weight is attached, is movable. In each block the pulleys are free to rotate on the same axle.

FIG. 70.

If there were no friction the tension of the string would be the same everywhere. We will at first neglect friction. Suppose the weight W raised a distance x , P will be lowered a distance nx . The total work

done is zero, and further, as in the previous example, and in all such cases where friction is non-existent, the work done by the tension on the lower block is equal to the work done on the tension by P , for there are n portions of the string pulling at the lower block, and the work done by these is $Tx \times n$, while the work done against T by P is $T \times nx$. These therefore cancel, and we have to consider only the work done by P and on W , and these are Pnx and Wx ;

$$\therefore Pnx = Wx;$$

$$\therefore W = nP.$$

In all such questions where the length of the connecting string remains unchanged and the tension is the same throughout, the amounts of work done by and against the tension cancel one another, and can be omitted altogether, and the same applies to rigid weightless connections.

We will illustrate by further examples of machines, but will first define some frequently used terms.

47. If a force P , applied to a machine, causes it to exert a force W , the ratio W/P is called the **Mechanical Advantage** of the machine. If, while P is displaced a distance x , W is displaced a distance y , the ratio x/y is called the **Displacement Ratio**, or frequently the **Velocity Ratio**.

If there is no friction and the connections are weightless, then, by the principle of work,

$$Px = Wy,$$

$$\frac{W}{P} = \frac{x}{y};$$

or, in this case, the mechanical advantage is equal to the displacement ratio.

If there is friction, Px will not be equal to Wy , but $Wy < Px$, or there will be less work done by the machine than is put into it, or energy is wasted by the machine. The ratio $\frac{Wy}{Px}$

is then called the **Efficiency**. For a frictionless weightless machine the efficiency is unity, and in no case can it be greater than unity.

The mechanical advantage can be anything from zero to infinity.

As a simple example take the system of pulleys already illustrated, calling W the total weight of the moving block and weight lifted. The displacement ratio is n , and therefore the mechanical advantage is n if there is no friction.

Suppose $n=4$, then a weight of 100 lbs. could be lifted by a force of 25 lbs. in the theoretical case of no friction. But in practice the force required to raise the weight of 100 lbs. will be greater than 25 lbs.-wt., say it was found to be 40 lbs.-wt. In lifting W 1 ft., 100 ft.-lbs. of work are done, P goes down 4 ft., and 160 ft.-lbs. of work are done by P . The ratio $\frac{100}{160} = \frac{5}{8}$ is the efficiency of the system. In other words $\frac{5}{8}$ of the work done on (or energy supplied to) the machine is utilized, the remaining $\frac{3}{8}$ being wasted.

48. The Wheel and Axle.

A wheel is capable of rotation about a horizontal axis, and a weight W to be raised is attached to one end of a rope wound round the axle, the other end of the rope being fixed to the axle. The force P is applied tangentially to the wheel.

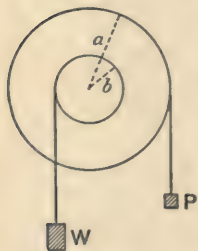


FIG. 71.

In this case it is best to consider a complete rotation of the wheel. If the radii of the wheel and axle are a and b , in one revolution of the wheel P descends a distance $2\pi a$, and W rises a distance $2\pi b$. The displacement ratio is therefore a/b .

49. The Differential Wheel and Axle.

Sometimes the above apparatus is modified into the differential wheel and axle, in which there are two portions of the axle of different radii b_1 and b_2 , and the weight, instead of being tied to the rope, is slung on it by a ring, so that the rope can unwind from the smaller axle, and wind up on the larger. In this case, in one rotation of the wheel the rope is shortened by $2\pi(b_1 - b_2)$, and regarding the ropes as vertical, the weight rises $\pi(b_1 - b_2)$; the displacement ratio is $\frac{2a}{b_1 - b_2}$.

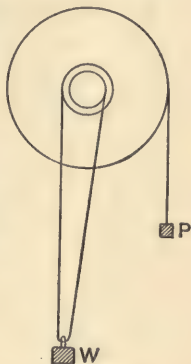


FIG. 72.

50. The Differential Pulley (Weston's).

A pulley is cut with grooves side by side, one a little larger than the other. An endless chain passes round the grooves

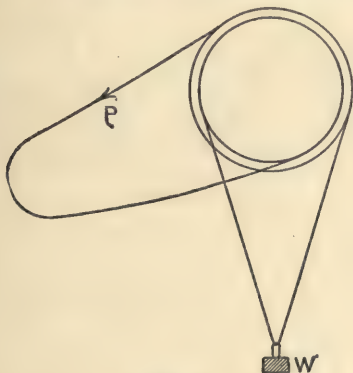


FIG. 73.

as in the figure. The chain is pulled by a force P , and prevented from slipping in the grooves by projections engaging the chain.

Example. If the outer and inner radii are a and b , prove that the displacement ratio is $\frac{2a}{a-b}$.

51. The Screw Press.

In a screw press a pressure is exerted by a plate A, which is moved by the rotation of the handle. In one rotation the

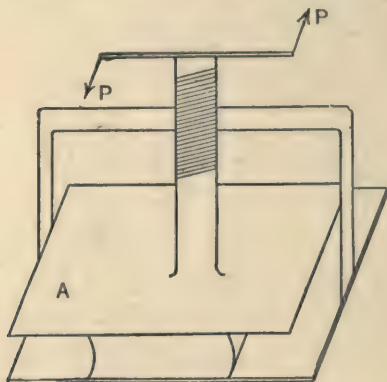


FIG. 74.



FIG. 75.

plate moves forward the distance between two successive portions of the thread, which distance is called the **Pitch** of the screw. If the pitch is p and the arm $2a$, and a force P is applied at each end of the arm, in one revolution the forces P each do work

$$P \cdot 2\pi a,$$

and work Wp is done against the resistance W ;

$\therefore$ neglecting friction,

$$P \cdot 4\pi a = Wp,$$

$$\frac{W}{P} = \frac{4\pi a}{p}.$$

Notice that in the above example a couple $P \cdot 2a$ is operating, and the work done by it as the arm rotates through four right angles is $2Pa \cdot 2\pi$. In fact, in any case, the work done

by a couple L , while its arm turns through an angle θ (in circular measure) is $L\theta$.

A **Lifting Jack** is of the same nature as the screw press, but the axis of the lever arm is distinct from and makes an angle with the axis of the screw, so that bevel wheels are required to cause one to actuate the other.

52. Cogged Wheels.

Suppose an arrangement as in the figure A and B, each consisting of two wheels fixed to the same axis. Suppose

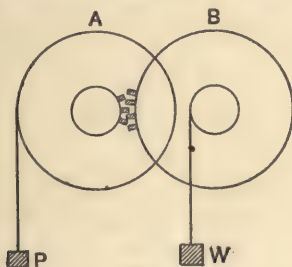


FIG. 76.

there are m teeth on A and n on B, and a and b are the radii of the wheels to which P and W are attached. A will make n revolutions, while B makes m .

Example. Prove that the displacement ratio is na/mb .

53. Work done by a Variable Force.

If the force, instead of being constant as in the above examples, varies as the point of application moves, we have generally to calculate the amount of work done by the Integral Calculus.

Graphically the work may be represented as follows :

Suppose the point of application of the force to move in a straight line, and the force to act in the direction of the displacement.

Draw a graph representing the displacement along the horizontal axis, and the force along the vertical (a P - s graph). Thus, in the figure, AC represents the *magnitude* of the force when the point of application is at A , M_1P_1 is the force when at M_1 , and so on.

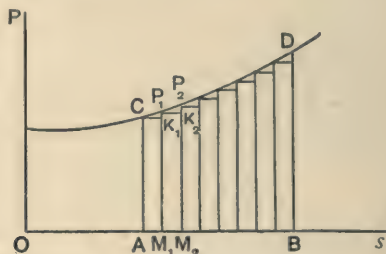


FIG. 77.—Force-space graph.

Suppose we want to find the work done as the point of application moves from A to B .

We will prove that the work is represented by the area $ABDCA$ between the graph, the axis of s , and the ordinates at the extreme positions A and B .

To prove this, divide the distance AB into a large number of short intervals by points, $M_1, M_2, M_3, \dots$, and draw ordinates $M_1P_1, M_2P_2, M_3P_3, \dots$, to meet the graph, and $CK_1, PK_2, \dots$, parallel to Os , as in the figure.

In any interval M_rM_{r+1} the force will vary only slightly if the interval is made short enough. Hence the work done in the distance AM_1 , which is the average force multiplied by the distance, is nearly equal to $CA \times AM_1$, or is nearly represented by the rectangle ACK_1M_1 .

So in the next interval the work is nearly represented by the rectangle $P_1M_1M_2K_2$ and so on, so that the whole work done will be represented nearly by the area bounded by CA, AB, BD , and the series of steps $CK_1P_1K_2P_2\dots$.

The shorter the little intervals (or the more numerous they

are) the more accurate will this representation become, and also the nearer will the series of steps be to coincidence with the graph itself.

Hence, when we make the number of intervals indefinitely large, we have the result that the work will be accurately represented by the area between the graph, the ordinates, and the axis of s .

The representation of the work by an area must be understood in the following way. Suppose on squared paper the side of a square in the horizontal direction represents a distance of one foot, and the same length in the vertical direction represents a force of 5 lbs.-wt., then a unit square will represent 5 ft.-lbs. of work, and similarly in any case.

54. Springs and Extensible Springs.

The only case we shall examine of work done by a variable force is where a spiral spring or an elastic string is stretched. We have already mentioned that when an elastic string is fixed at one end and a weight is hung at the other, the string is found to increase in length, and the extension of the string is proportional to the weight applied, that is, to the stretching force or the tension of the string. This is a case of **Hooke's Law**. The extension for different strings of the same material is also found to be proportional to the length of the string and inversely proportional to its cross section.

Thus the extension is proportional to $\frac{Pl}{A}$, where P is the force applied. A is the area of cross section, l the length when unstretched.

If we call s the extension, we can therefore write

$$s = \frac{Pl}{EA},$$

where E is a constant for the material and is called **Young's Modulus**.

Writing this as

$$P = \frac{EA}{l} s,$$

we may also write it

$$P = \lambda s/l$$

or

$$P = ks,$$

where $\lambda = EA$, and is frequently called the *Modulus of the String*, and $k = EA/l$.

When $A = 1$ and $s = l$ we find that $P = E$, or E is numerically equal to the force which would stretch a string of unit cross section to double its natural length, if it did not break previously, or Hooke's Law cease to hold.

The extension of a spiral spring follows the same law in so far as it is proportional to the force applied, but we may have either extension or compression in the spiral spring, whereas we only get extension in a string.

55. Work done in Stretching a String or Spring.

Since the force acting when the string is elongated an amount s above its natural length is ks , if we draw the force-

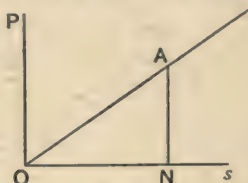


FIG. 78.

space graph it is a straight line, and the work done in stretching to a length s , where $ON = s$ and $AN = ks$, is

$$\frac{1}{2} ON \cdot AN = \frac{1}{2} ks^2 = \frac{1}{2} Ps,$$

if P is the final tension.

This result applies to all cases of strings and springs, and in springs both to compression and elongation. For the spring $P = ks$, but k cannot be written as EA/l ; k may be called the stiffness of the spring.

56. Roberval's Balance.

Roberval's Balance is very extensively used for small weights such as letters and for household purposes, such as for weights up to 7 lbs., where great accuracy is not required.

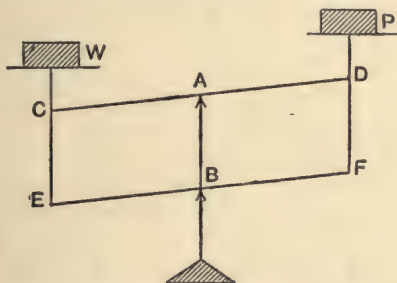


FIG. 79.—Roberval's Balance.

It consists of a framework of four jointed rods, of which CD, EF are capable of rotating about fixed knife-edges at A and B. As the balance moves the bars continue to form a parallelogram, CE and DF, which carry the scale-pans always remaining vertical, and consequently the scale-pans move through equal distances. Hence, if there is equilibrium when weights P and W are in the scale-pans, the principle of work shews that

$$P = W.$$

No work is done by or against gravity by the framework as a whole, for any portion of one side rises exactly the same amount as the corresponding portion of the other side falls.

It should be noticed that the weights may be placed anywhere in the pans. This condition is necessary in any weighing apparatus. In the common balance, if the weights are not placed in the centre of the pan, the pan simply moves so that the centre of mass of the scale-pan and weight comes vertically under the point of suspension, and consequently the position of the weight in the scale-pan is immaterial.

57. Weighing Machine.

A simple form of weighing machine or weigh-bridge consists of a series of jointed bars as shewn in the figure.

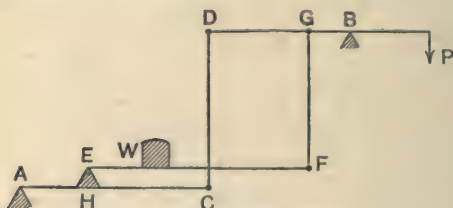


FIG. 80.—Quintenz's Balance.

A, B are fixed fulcrums about which the bars can rotate.

EF is the platform on which the body to be weighed is placed.

C, D, F, G are joints.

E rests on a fulcrum fixed to AC.

Now, if the bars move, C and D will move the same distance, say x downwards.

E and H will move a distance $\frac{AH}{AC} x$.

F and G will move a distance $\frac{BG}{BD} x$.

Therefore if $\frac{AH}{AC} = \frac{BG}{BD}$, E and F will move the same distance, and therefore the whole platform moves the same distance.

If we call the distance the platform moves x' ,

G moves down a distance x' ,

K moves up a distance $\frac{KB}{BG} x'$,

and therefore if a weight P at K is required to balance W,

$$P \frac{KB}{BG} x' = W x',$$

$$W = \frac{KB}{BG} P.$$

As the whole platform moves the same distance, it does not matter on what portion of it the load is placed.

Often this balance is made with $KB = 10 BG$, and is then called a Decimal Balance.

The work done by the parts of the machine is not taken into account in the above theory.

EXAMPLES.

1. In the system of pulleys in which a single string passes round the two blocks, there are six portions of the string between the two. What is the displacement ratio? If the efficiency of the apparatus is 30 per cent., what energy has to be supplied to raise a weight of 150 lbs. 20 ft.?

2. If the lower block contains two pulleys and weighs 2 lbs. and the upper block contains three, what would the mechanical advantage be in raising a 20 lb. weight if there were no friction, and what energy is wasted in raising it 10 ft. if a force of 6 lbs. is actually required?

3. A fixed pulley and a single movable one are used for raising a weight of 200 lbs. If the efficiency is 64 per cent., what force will be exerted, and how much energy is expended in raising the weight 20 ft.?

4. A man suspends himself by standing in a bucket at one end of a rope, and pulling at the other end. If W is the weight of the man, and W' of the bucket, find the tension of the rope and the pressure of the man on the bucket.

5. Two painters seated on a plank haul themselves up by means of a block and tackle at each end of the plank. If each man weighs 10 stone, and the plank and lower blocks weigh a cwt., and there are three sheaves in each block, find the pull each man must exert, and his pressure on the plank, neglecting friction.

6. The handle of a winch has a radius of 18 ins., and the drum is 6 ins. in diameter. Find the force on the handle necessary to give a pull of 2 cwt. on the rope. Explain how gear wheels may be used, so that the same force on the handle would give a pull of half a ton.

7. A bucket of weight 50 lbs. is raised by a windlass whose dimensions are as in Question 6. What force would be required if there were no friction? What is the efficiency if the force actually required is 12 lbs.-wt.?

8. An ordinary winch is used to lift a weight of 112 lbs. The length of the handle is 20 ins., and the diameter of the barrel of the winch is 8 ins. What force on the handle would be required to raise the weight if there were no friction?

If a force of 30 lbs. is actually required, find the energy wasted on account of friction in raising the weight 20 ft.

9. In a differential wheel and axle the two parts of the axle are of radii 4 and 5 ins., and the force is applied at the end of an arm 18 ins. long. To raise a weight of 2000 lbs. a force of 96 lbs. is required. What is the efficiency?

10. In a differential wheel and axle the two parts of the rope on which the weight W is slung make an angle 2α with one another, and a, b, r are the radii of the two parts of the axle and of the wheel. Prove that

$$Pr = \frac{1}{2}W(b - a) \sec \alpha,$$

neglecting friction.

11. The handle of a screw press is 20 ins. long, and the screw advances $\frac{1}{4}$ in. in each complete revolution. Find the pressure exerted by the press when a force of 12 lbs.-wt. is applied at each end of the handle. Find also the work done in a complete revolution of the handle (neglect friction).

12. What will the pitch of the screw press be if the handle is 2 ft. long, and a force of 10 lbs.-wt. applied at each end of the handle causes a pressure of 5000 lbs.-wt. to be exerted by the press (neglect friction)?

13. Two screws of pitches p and p' are cut on the same cylinder. The first works in a fixed nut, the second on a nut that can move longitudinally, but is prevented from turning. The driving force is applied at the end of an arm of length b ; the resistance acts on the movable nut. Find the ratio of driving force to resistance.

14. A body is pulled up a rough incline AB by a force P parallel to the plane. Shew that the work done is equal to the sum of the work done in pulling the body by a horizontal force along an equally rough horizontal plane equal to the base AC , and then lifting it vertically from C to B .

15. A weight of 260 lbs. is pulled up a plane of length 13 ft. and height 5 ft. Find the force exerted and the work done if the coefficient of friction is 0.5

- (i) when the force is parallel to the plane;
- (ii) when it is horizontal;
- (iii) when the least possible force is used (see Art. 22, Ex. 5).

CHAPTER V.

ANY FORCES IN ONE PLANE.

58. Couples.

We have seen that two equal, parallel and opposite forces cannot be reduced to a single force. We therefore speak of two such forces as forming a **Couple**.

The distance between the two forces is called the **Arm** of the couple and the product Pa of either force into the arm of the couple is called the **Moment** or **Torque of the Couple**.

We shall find that the moment is the essential thing connected with a couple, neither direction nor magnitude of the forces nor the length of the arm separately mattering, the only thing by which different couples (in a plane) are distinguishable being their moment.

The effect of a couple on a body may be seen by placing a rod on a horizontal table and applying two forces by means of spring balances to any points in the rod. If the rod is held fixed until the forces are made equal, besides being parallel and opposite, and then released, the rod will always begin to rotate about its centre at whatever points the forces are applied, and the magnitude of the effect produced depends on the moment of the couple. Further, two forces, as in



FIG. 81.—A couple.

Fig. 82, will make the body rotate in the positive direction of trigonometry, while those in Fig. 83 will make it rotate in the opposite direction. We consequently distinguish between positive and negative moments. As usual it does not matter

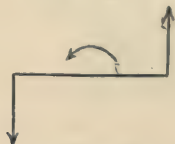


FIG. 82.

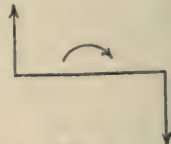


FIG. 83.

which direction of rotation we call the positive one if we are consistent. The direction of the moment of a couple may be represented by a curved arrow as in the figures. The curved arrow may also be used for the moment of a single force about a point.

59. Theorems on Couples.

The algebraic sum of the moments of the forces of a couple about any point in its plane is constant.

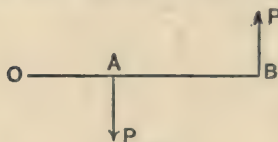


FIG. 84.

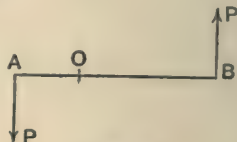


FIG. 85.

Thus, if O is any point, the sum of the moments about O in Fig. 84 is

$$\begin{aligned} & P \cdot OB - P \cdot OA \\ &= P \cdot AB, \end{aligned}$$

and in Fig. 85 the sum of the moments is

$$\begin{aligned} & P \cdot OA + P \cdot AB \\ &= P \cdot AB. \end{aligned}$$

Thus the algebraic sum of the moments of a couple about any point is what we have defined as the moment of the couple.

As a special case it may be noticed that the moment of a couple is equal to the moment of one of the forces about any point in the line of action of the other, which is also only a statement of the definition of the moment of a couple.

Also it may be noticed that two equal and opposite forces in the same straight line may be said to form a couple of zero moment, and conversely the moment Pa of a couple can only be zero if either

$$a=0,$$

which means that the forces are in the same straight line, and are consequently in equilibrium, or if

$$P=0,$$

in which case there are no forces at all.

60. Composition of Couples.

Two couples in the same plane are equivalent to one couple whose moment is the sum of the moments of the separate couples.

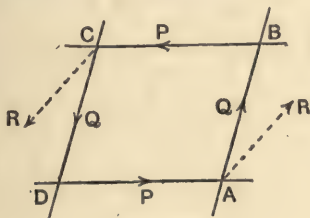


FIG. 86.

Case (i). When the lines of the forces are not all parallel.

Let the forces P, P, Q, Q , form two couples, the lines of action of the forces intersecting at A, B, C, D .

The forces P, Q , at A can be combined into a single force R , and so can the P and Q at C . These single forces will be equal, parallel and opposite, for they are both resultants of P and Q at the same angle. Hence the two couples are equivalent to a single couple.

Also the moment of this resultant couple

=moment about C of R at A

=sum of moments about C of P and Q at A (Art. 30)

=sum of moments of the original couples.

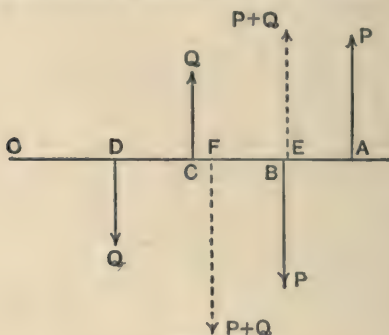


FIG. 87.

Case (ii). When the forces of the two couples are parallel.

In Fig. 87, P and Q at A and C are equivalent to a single force $P+Q$ at some point E and parallel to P and Q, and similarly for P and Q at B and D. Hence the couples are equivalent to one couple.

Also the moment of the resultant couple

=sum of the moments about any point O of the two forces $P+Q$ at E and F.

But the moment of $P+Q$ at E

=sum of moments about O of P at A and Q at C,

and similarly for the moment of $P+Q$ at F.

Hence the moment of the resultant couple

=algebraic sum of moments of the four original forces

=sum of moments of the two original couples.

Hence two couples of equal and opposite moments in a plane balance. For they are equivalent to a single couple

of zero moment, which we have seen (Art. 59) is a case of equilibrium.

61. By reversing the forces of one couple we have the result that :

Any two couples of equal moment in the same direction are equivalent to one another.

Thus, two couples such as shewn in Fig. 88 are exactly equivalent in their effects on a body.

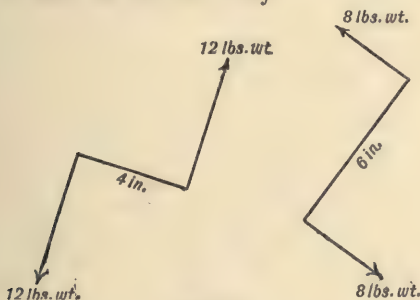


FIG. 88.

It follows, as we have stated already, that the moment is the only quantity of importance about a couple, neither position, magnitude or direction of the forces, nor the arm of the couple being separately of consequence.

In other words the forces of a couple can be moved to any part of its plane, turned round through any angle, and altered in magnitude to any extent, provided only that they are kept equal and parallel so as still to form a couple, and that the arm is correspondingly altered to keep the moment unchanged. The effect on a rigid body on which they act in both positions of the forces will be unaffected.

From the case of two couples it follows at once that any number of couples in a plane can be compounded into one couple whose moment is the algebraic sum of the moments of the separate couples.

62. Couples in Parallel Planes.

Though it really belongs to three-dimensional work, we will here prove that a couple may be moved from one plane into any parallel plane without altering the effect on a rigid body on which it acts.

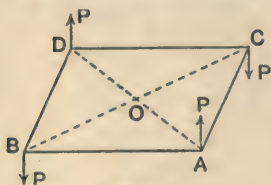


FIG. 89.

We will first shew that two equal and opposite couples in parallel planes balance.

The forces of one couple may be altered so as to make them equal and parallel to those of the other. The arms will then also be equal.

Let the P 's at A and B represent one couple, and the P 's at C and D represent another of equal and opposite moment, all the forces being parallel and AB parallel to CD .

Then $ABDC$ is a parallelogram, and the forces at A and D are equivalent to $2P$ at its centre O , and the forces at B and C are equivalent to $2P$ at O in the opposite direction. Hence the four forces balance, or the original couples balance.

Hence two equal couples in the same direction are equivalent.

Examples. 1. Forces 2, 3, 2, 3 lbs.-wt. act along the sides AB , etc., of a rectangle $ABCD$, in which AB is 5 ft. and BC 4 ft. Find the resultant couple.

2. Solve the same problem as in Example 1, but with $ABCD$ a parallelogram, whose angle ABC is 120° , the lengths of the sides being as before.

3. Forces 1, -2, 3, 1, -2, 3 lbs.-wt. act along the sides taken in order of a regular hexagon of side 2 ft. What is the resultant couple?

63. We have seen that for the composition of forces the parallelogram law is required. For couples in one plane we require only simple addition of the moments. We shall, however, see later that in dealing with couples in different non-parallel planes the parallelogram law is required also.

64. Composition of a Force and Couple.

We can now see how to compound a force and a couple in one plane.

Let P be a force at O , and let a couple be formed by two forces Q acting at A and B . We can replace this couple by

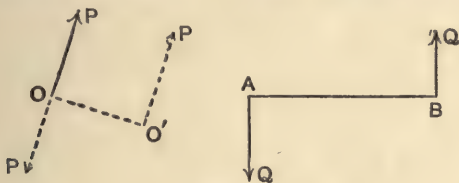


FIG. 90.

any other of the same moment. Thus we can shift the couple, turn round the forces, and alter their magnitudes until they are each P , and one of them acts at O in the opposite direction to the P already acting there. It is necessary that OO' , the arm of the new couple, should equal $\frac{Qa}{P}$ in order to keep the moment unchanged. We now have the two forces at O balancing one another, and are left with a single force P at O' .

Hence a couple of moment L and a force P compound into a single force P at a distance L/P from the original force. The side of the original force on which the resultant lies depends on the direction of the couple.

Conversely, a single force may be replaced by an equal force acting at any chosen point and a couple.

For in Fig. 91, if we have a single force at A to begin with, we can introduce two opposite forces at the chosen point O equal and parallel to the original force ;

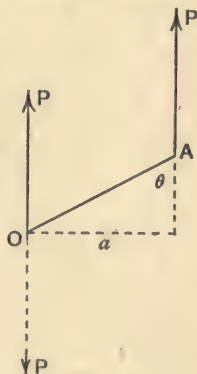


FIG. 91.

and now one force at O and the force at A form a couple whose moment is Pa , if a is the perpendicular from O on the line of action of P . Thus we are left with a force at O equal and parallel to the original force, and a couple.

The moment of the couple may be written $P \cdot OA \sin \theta$, where θ is the angle between P and OA .

Examples. 1. Forces 2, 3, 2 act along the sides AB , BC , CD of a rectangle, in which AB is 5 ins. and BC 4 ins. Find the resultant force in direction and magnitude, and find where its line of action cuts AB .

2. Forces are represented in magnitude and position by AB , CD , BD in a square $ABCD$. Find the resultant in magnitude and position.

3. $ABCD$ is a parallelogram in which AB is 6 ins., BC is 4 ins. and the angle A is 60° . Forces 5, 3, 5 lbs.-wt. act along CD , DA , AB . Find where the resultant force cuts AB .

65. Composition of any Forces in one Plane.

We can now find an easy way of simplifying any number of forces in a plane.

Suppose there are any forces P , Q , R acting at A , B , C , etc.

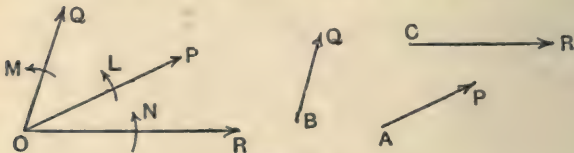


FIG. 92.

We can take any point O and replace P by an equal force at O and a couple whose moment is L , say. Similarly Q is replaced by Q at O and at couple M , and so on. In the figure the couples are represented by the curved arrows with the magnitudes of their moments against them. Notice in the figure N will be negative if L and M are positive.

We have now a number of forces at O , and a number of couples. We can reduce these by the former methods. Thus

the forces at O can be reduced to a single force and the couples reduce to one couple. Either the single force or the couple or both may happen to be zero.

Hence we are left with

(i) a single force or (ii) zero,

together with

(iii) a couple or (iv) zero.

Now, if we have (i) and (iii), we can further reduce them to a single force :

(i) and (iv) also give a single force,

(ii) and (iii) give a couple,

(ii) and (iv) give zero.

Thus we have the result that any system of forces whatever in a plane can be reduced to either

(a) a single force,

(b) a couple,

(c) zero.

66. It should be noticed that in reducing any system of forces to a simpler system there is no change in the sum of the resolutes in any given direction, nor is there any change in the sum of the moments about any given point.

For, first, the sum of the resolutes of the forces of a couple in any direction is evidently zero, and therefore when a force is replaced by a parallel and equal force and a couple, the sum of the resolutes is unaffected. Also in combining the forces into a single force (by Art. 16), the sum of the resolutes is unaffected, and so it is again in finally compounding a force and a couple into a single force.

Nor secondly, is the moment about any given point affected, for the replacing of a force by a force and a couple is performed by the addition of two equal and opposite forces in the same straight line, and the total moment is left unaffected, and the

same is true (by Art. 30) in combining forces by the parallelogram law, and obviously also in combining couples by the simple addition of their moments.

Thus, in other words we may say when we reduce the forces to a single resultant force, the resolute of the resultant in any direction is equal to the sum of the resolutes of the original forces in the same direction, and the moment of the resultant about any point is equal to the sum of the moments of the original forces about the same point.

When a single force at A is replaced by a force and a couple, the force is the same wherever O may be, but the couple varies according to the point O chosen. The same will apply to the reduction of any number of forces to a single force at O and a couple. The single force at O is the same wherever O may be, but the couple depends on the position of O.

Examples. 1. Reduce the system of forces shewn in the diagram to (i) a force at A and a couple, (ii) a force at B and a couple, (iii) a single resultant, finding the position of the latter.

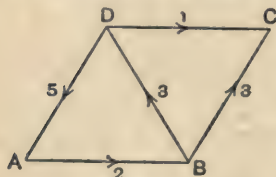


FIG. 93.

The figure consists of two equilateral triangles.

Let a = the length of each side,

p = the perpendicular distance between two opposite sides of the parallelogram, so that $p = \frac{a\sqrt{3}}{2}$.

(i) Replace each force by a force at A and the appropriate couple; we then have forces and couples as in (a).

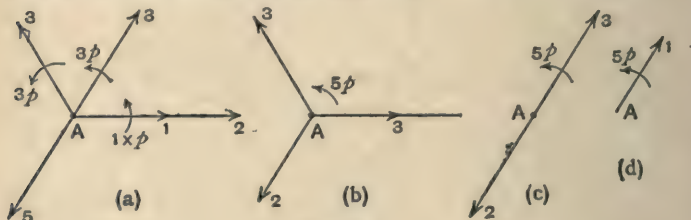


FIG. 94.

For example, the force 3 along BC is equivalent to 3 at A and a couple $3p$ in the direction marked.

These forces and couples compound next into the forces and couples shewn in (b). Now the two 3's at an angle 120° are equivalent to 3 along the bisector of the angle between them; thus we get the forces as in (c), and finally as in (d).

Thus we are left finally with a force 1 at A and a couple $5p$ or $\frac{5a\sqrt{3}}{2}$.

(ii) If we replace the forces by forces at B and couples, we have in the same way the following successive steps :

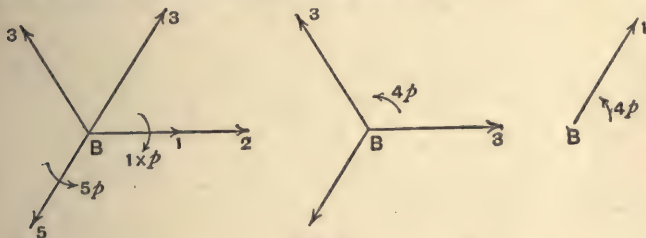


FIG. 95.

(iii) The single resultant is 1 and parallel to BC, and its position is got by noting that its distance from A will be $5p$, for the force at A and couple can be represented as

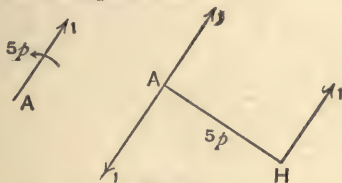


FIG. 96.

and we are left with a single resultant 1 at a point H, where $AH = 5p$.

We can otherwise get the position of the resultant in the following way :

If X is a point on the line of action of the resultant, the sum of the moments of the original forces about X must be zero, since the moment of the resultant about X is zero. Suppose that the resultant cuts AB at a point X at a distance x from A. We write down that the sum of the moments about X is zero.

$$\therefore 3(a-x) \frac{\sqrt{3}}{2} + 3(a-x) \frac{\sqrt{3}}{2} - 1 \times \frac{a\sqrt{3}}{2} + 5 \times \frac{x\sqrt{3}}{2} = 0 ;$$

$$\therefore 3a - 3x + 3a - 3x - a + 5x = 0 ; \quad \therefore x = 5a,$$

thus giving a point on the line of action of the resultant. It will be found that this agrees with the result obtained above, that the resultant acts at a distance $5p$ from A.

2. Shew that if three forces act along the sides of a triangle and are proportional to the sides, they are equivalent to a couple.

Let the forces be ka , kb , kc .

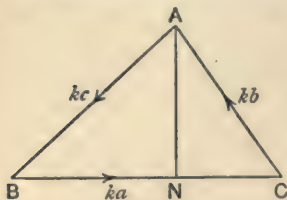


FIG. 97.

Replace ka by an equal force at A and a couple. We now have three forces at A parallel to the three sides of the triangle taken in order and proportional to them, and also a couple. The forces at A are in equilibrium by the triangle of forces, and therefore the original forces reduce to a couple whose moment is

$$ka \times AN = 2k \times \text{area of the triangle.}$$

3. Shew that forces represented in direction, magnitude and position by the sides of a polygon, taken in order, are equivalent to a couple.

4. Shew that if the sums of the moments of a system of forces about three points not in the same straight line are the same, the forces must be equivalent to a couple.

5. Shew that if the sums of the moments of a system of forces are the same about two different points, the forces are either equivalent to a couple or have a resultant parallel to the line joining the two points.

6. In each of the following diagrams (each of which is a square of side 5 in.) reduce the forces to

- (i) a single force at A and a couple,
- (ii) " " B " "
- (iii) " " O " "

and when reduced to a single resultant, find where it cuts AB

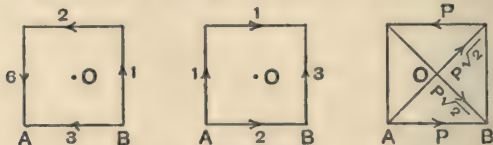


FIG. 98.

7. Forces -2 , 1 , 4 , 4 act along the sides AB, etc., of a rectangle ABCD, where $AB = 2BC$. Find the magnitude, direction and position of the resultant.

8. Forces P , $2P$, $-P$, $2P$ act along the sides AB , etc., of a square $ABCD$, and $P\sqrt{2}$ along each of BD and CA . Shew that the forces reduce to a couple, and find its moment.

67. Conditions for Equilibrium.

We have seen that any system of forces in a plane can be reduced either to

- (a) a single force,
- or (b) a couple,
- or (c) zero.

In the latter case the forces are in equilibrium, and it will be seen (from Art. 65) that for this to occur, the single force and couple to which the forces were first reduced must be both zero.

We can now examine the conditions necessary that any system of forces in a plane may be in equilibrium.

It will be found that when expressed algebraically there are three conditions necessary for equilibrium, but that the conditions may be expressed in various alternative ways.

The most usual form is as follows :

A system of forces in a plane is in equilibrium if the sums of the resolved parts of the forces in any two different directions are separately zero, and the sum of the moments about one point is zero.

That these conditions are sufficient to secure equilibrium may be seen in the following way. By Art. 65 the forces must reduce either to

- (a) a single force,
- (b) a couple,
- (c) nothing at all.

If the above conditions hold we have

(1) The forces cannot reduce to a single force, for if the sums of the resolved parts in any two directions are separately zero, the resolved part of the resultant would be zero in each of the two directions, which is impossible.

(2) The forces cannot reduce to a couple, for a couple has a definite moment, not zero, about every point in its plane.

$\therefore$ the forces must reduce to nothing at all.

68. Other forms of putting the conditions for equilibrium are the following :

(1) The sums of the moments about two points A and B are separately zero, and the sum of the resolved parts in a direction not perpendicular to AB is zero.

(2) The sums of the moments about three points not in the same line are separately zero.

We will demonstrate the latter and leave the remaining case to the student.

If the sum of the moments about A is zero, the forces cannot reduce to a couple, but may reduce to a single force through A. Similarly the other conditions give only a possible resultant through B and through C. But if A, B, C are not in one straight line the resultant force cannot pass through the three, and therefore there cannot be a resultant force at all.

69. Solution of Examples.

In solving examples the method nearly always to be followed will be to first draw a diagram and mark all the forces, then write down the three equations obtained by

(i) Equating the sums of the resolved parts in two different directions to zero separately.

(ii) Equating the sum of the moments about a chosen point to zero.

From the three equations thus written down the unknown quantities are to be found, and therefore must not be more than three if the question is capable of solution.

In selecting a point O to take moments about, it may be noticed that if one of the forces passes through O that force will have no moment about O, and therefore will not appear

in the equation of moments. If the point of intersection of two of the forces is chosen, both these forces will be excluded from the equation.

Examples. 1. A rod, of weight 10 lbs., is supported by a hinge at A, and carries a weight of 20 lbs. at B. It is tied back so as to make an angle 60° with the upward vertical by a horizontal string attached at B. Find the tension of the string and the reaction at A.

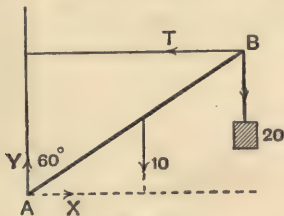


FIG. 99.

Here the direction of the reaction at A is at first unknown, and it is simplest to represent it by horizontal and vertical components X and Y.

Let $2a$ be the length of the rod.

We then have,

resolving horizontally, $T = X$,

„ vertically, $Y = 10 + 20 = 30$;

moments about A, $T \times AC = 20 \times BC + 10 \times \frac{1}{2}BC$

or $T \cdot 2a \cos 60^\circ = 20 \times 2a \sin 60^\circ + 10a \sin 60^\circ$;

$$\therefore T = 25\sqrt{3} = 43.3 \text{ lbs.-wt.,}$$

$$X = 25\sqrt{3}, \quad Y = 30.$$

The resultant reaction

$$= \sqrt{X^2 + Y^2} = \sqrt{1875 + 900} = \sqrt{2775} = 52.7 \text{ lbs.-wt.,}$$

and if θ is the angle the resultant makes with the horizontal,

$$\tan \theta = Y/X = \frac{6}{5\sqrt{3}} = \frac{2\sqrt{3}}{5}; \quad \therefore \theta = 34^\circ 43'.$$

2. A ladder rests against a smooth vertical wall with the lower end on a horizontal floor, the coefficient of friction at the floor being 0.5. Find the greatest angle the ladder can make with the vertical. Find also the greatest angle if the wall were not smooth, but there were a coefficient of friction of 0.4 at the upper end.

Case (i). When the wall is smooth, the forces being as in Fig. 100.

Let $2a$ = the length of the ladder
and W = the weight of the ladder.

Resolving vertically, $R = W$;

„ horizontally, $R/2 - S = 0$;

moments about A, $S \times BC - W \times AC/2 = 0$

or $S \times 2a \sin \theta - Wa \cos \theta = 0$.

Hence

$$R = W,$$

$$S = W/2;$$

$$\therefore W \sin \theta = W \cos \theta;$$

$$\therefore \tan \theta = 1;$$

$$\therefore \theta = 45^\circ.$$

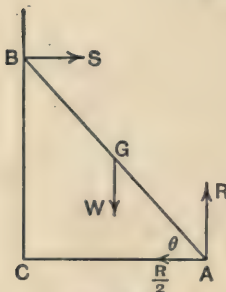


FIG. 100.

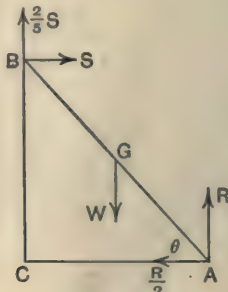


FIG. 101.

Case (ii). When there is friction at the wall with a coefficient 0.4, the forces are as in Fig. 101.

Resolving vertically, $R + \frac{2}{3}S = W$;(1)

„ horizontally, $S - \frac{R}{2} = 0$;(2)

moments about A,

$$S \cdot 2a \sin \theta + \frac{2}{3}S \cdot 2a \cos \theta = Wa \cos \theta. \text{(3)}$$

From (1) and (2),

$$S = \frac{5}{12}W,$$

$$R = \frac{5}{6}W.$$

Substitute in (3), $\frac{5}{6} \sin \theta + \frac{1}{3} \cos \theta = \cos \theta$;

$$\therefore \frac{5}{6} \sin \theta = \frac{2}{3} \cos \theta;$$

$$\therefore \tan \theta = \frac{4}{5};$$

$$\therefore \theta = 38^\circ 40'.$$

3. A thin rod of length $2l$ passes over a horizontal peg, and one end A presses against a vertical wall at a distance a from the peg. If there is no friction at the peg, what is the least coefficient of friction at the wall if the rod makes an angle θ with the horizontal?

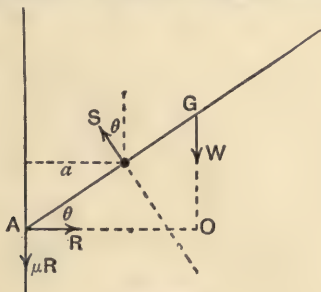


FIG. 102.

(i) Suppose the end A on the point of moving upwards, the forces are then as in the figure, and we have

$$\text{horizontally,} \quad R = S \sin \theta; \dots\dots\dots(1)$$

$$\text{vertically,} \quad S \cos \theta = W + \mu R; \dots\dots\dots(2)$$

$$\text{moments about A,} \quad S \cdot AD = W \cdot OA,$$

$$\text{or} \quad Sa \sec \theta = Wl \cos \theta. \dots\dots\dots(3)$$

$$\text{From (3),} \quad S = W \frac{l}{a} \cos^2 \theta.$$

$$\text{,, (1),} \quad R = W \frac{l}{a} \cos^2 \theta \sin \theta.$$

$$\text{,, (2),} \quad \mu R = W \frac{l}{a} \cos^3 \theta - W;$$

$$\therefore \mu = \frac{l \cos^3 \theta - a}{l \cos^2 \theta \sin \theta}.$$

(ii) If the end A was on the point of motion downwards, the friction is upwards, and it is only necessary to change the sign of R in (2); hence we get

$$\mu = \frac{a - l \cos^3 \theta}{l \cos^2 \theta \sin \theta}.$$

Hence the results shew that

if $a = l \cos^3 \theta$, the rod will remain at rest without any friction at A,

if $a < l \cos^3 \theta$, there is friction at A, acting downwards, keeping the end A from moving up,

if $a > l \cos^3 \theta$, there is friction at A, acting upwards, keeping the end A from moving down.

4. A full reel of cotton of radius a is kept at rest with the axis horizontal on a plane of inclination α , by holding the thread, which comes off horizontally and tangentially from the middle of the reel. If the weight is W , find the tension and friction, and find the least coefficient of friction for which the reel will rest in this way.

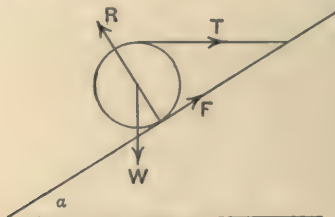


FIG. 103.

The forces are as in the figure.

Resolving parallel to the plane,

$$F + T \cos \alpha = W \sin \alpha ; \dots\dots\dots(1)$$

resolving perpendicular to the plane,

$$R = W \cos \alpha + T \sin \alpha ; \dots\dots\dots(2)$$

moments about the centre,

$$Fa - Ta = 0 ; \dots\dots\dots(3)$$

$$\therefore F = T ;$$

$\therefore$ by (1),

$$F = \frac{W \sin \alpha}{1 + \cos \alpha} = W \tan \frac{\alpha}{2} ;$$

also by (2),

$$R = W \cos \alpha + \frac{W \sin^2 \alpha}{1 + \cos \alpha} = W ;$$

$$\therefore F/R = \tan \frac{\alpha}{2},$$

or the coefficient of friction must be at least as great as $\tan \frac{\alpha}{2}$.

70. Use of the Angle of Friction.

Questions involving friction may often be attacked usefully by means of the angle of friction.

Thus, suppose AB is a ladder (not necessarily in limiting equilibrium) and the only forces on it are its weight W , the weight W' of a man on it, and the reactions at A and B . Then, if λ is the angle of friction at A , the normal reaction and

friction there can be combined into a single force, which makes an angle not greater than λ with the vertical. Similarly, if λ' is the angle of friction at B, the resultant reaction there makes an angle not greater than λ' with the horizontal. Draw APQ at an angle λ with the normal AMN, and BQN at an angle λ' with the normal BPM. Then the resultant

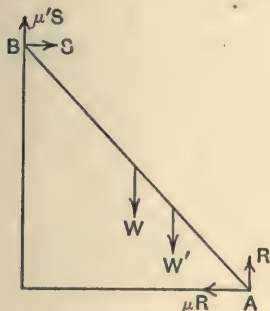


FIG. 104.

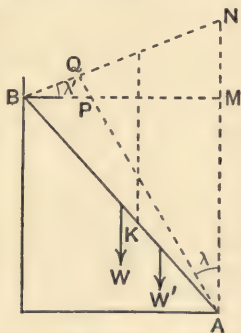


FIG. 105.

reactions at A and B must both pass through the quadrilateral MNQP or along its sides. Let also the weights W, W' be replaced by their resultant acting at K, say. We have now reduced the forces to three, and these three must meet in a point if the ladder remains at rest. This condition requires that the vertical through K should cut the quadrilateral PQMN.

Limiting equilibrium will occur when the vertical through K passes through Q. If it passes to the left of Q sufficient friction cannot be brought into play at A, and the ladder falls.

If the said vertical passes through the quadrilateral the ladder will remain at rest or sufficient friction will be brought into play to hold it up, but it will not be in limiting equilibrium. Further, it will be noticed that the necessary friction can be supplied in an infinite number of ways. For if we take any

point X in $PQNM$, and on the vertical through K , we would have equilibrium if the resultant reactions at A and B passed through X , and given the position of X we know the directions

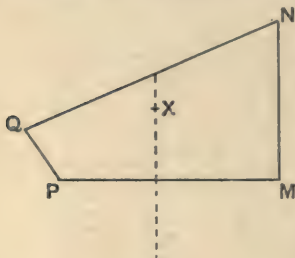


FIG. 106.

of these resultant reactions, and knowing the weight $W + W'$, we could find the reactions and their normal components and the friction at A and B . But X can be taken in an infinite number of positions. Hence we have an infinite number of solutions for the normal reactions and frictions.

In such a case the problem of finding the forces is said to be

indeterminate. If we proceed algebraically (and are not told that the friction is limiting) it will be found that there are four unknowns—the normal reaction and friction at each end. But we cannot form more than three independent equations; hence the equations are indeterminate, that is, have in general an infinite number of solutions.

Examples. 1. A rod AB , weighing 10 lbs., is supported by a smooth hinge at A , and has a weight of 10 lbs. hanging at B . It is kept at an angle θ with the downward vertical by means of a horizontal force P at B . Find P and the components of the reaction at A , when

(i) $\theta = 45^\circ$; (ii) $\theta = 30^\circ$.

Find also the resultant reaction at A in direction and magnitude in each case.

2. A rod AB , of weight 10 lbs., is supported by a hinge at A , and carries a weight of 30 lbs. at B . It is tied back by a string attached at a point C in the rod, to a point D vertically above A . If $AB = 3AC$, find the tension and reaction if the rod and string both make angles 45° with the vertical.

3. A rod AB , of weight 10 lbs. and carrying 20 lbs. at B , is supported by a hinge at A and by a string attached at B , and to a point vertically above A . If the rod makes an angle 60° with the downward vertical, shew that the shorter the string the greater the

tension. If the string cannot bear a tension of more than $34.6 (=20\sqrt{3})$ lbs.-wt., find the largest angle it can make with the vertical.

4. A rod AB , of weight W , is supported by a hinge at A , and carries a weight W' at B . It is tied back by a string attached at a point C to the rod, and to a point D in the vertical through A . If the string makes an angle θ with the vertical, and the rod makes an angle ϕ , and $AB=2a$, $AD=b$, find the tension of the string and the components of the reaction at A .

5. If in Question 4 ϕ is supposed given, what angle would the string require to make with the vertical so that there should be no vertical reaction at A (in other words, that the end A could simply rest against a vertical wall without friction) ?

6. A beam AB , of weight 4 cwt., is supported at A and carries a weight of 2 tons at B . The beam is held by a rope attached at B and to a point vertically above A . If the beam makes an angle 40° with the upward vertical, and the rope makes an angle 70° with the beam, find the tension of the rope.

7. A ladder 20 ft. long, weighing 100 lbs., rests with the upper end against a smooth vertical wall, and the lower on a smooth floor and 8 ft. from the wall. If it is kept from slipping by a horizontal rope attached to the lower end, find the tension of this rope.

8. A ladder 15 ft. long, weighing 100 lbs., rests against a smooth wall and on a smooth horizontal plane, being kept from slipping by a horizontal rope 9 ft. long attached to the lower end and to the wall. Find the tension of the rope when a man weighing 150 lbs. is (i) half way up, (ii) at the top.

9. A ladder of weight 100 lbs. rests against a smooth vertical wall, with the lower end on a horizontal floor, where the coefficient of friction is 0.4. If it is in limiting equilibrium, find the angle it makes with the horizontal.

10. If the ladder in Question 9 is 15 ft. long and is placed at an angle 60° with the horizontal, how high can a man of weight 150 lbs. climb it without its slipping ?

11. A ladder rests against a vertical wall and on a horizontal floor, the coefficients of friction being respectively 0.5 and 0.4. Find the limiting position of equilibrium.

12. A ladder rests at an angle 45° in limiting equilibrium against a vertical wall and a horizontal plane. If the coefficient of friction is the same at both ends, find it.

13. A ladder rests in limiting equilibrium at an angle θ with the horizontal against a vertical wall. If the coefficients of friction at the upper and lower ends are respectively μ and μ' , prove that

$$\tan \theta = \frac{1 - \mu\mu'}{2\mu'}.$$

14. A ladder of weight 100 lbs. rests between a smooth wall and smooth floor, being prevented from slipping by a rope attached to the foot of the wall, and to a point in the ladder. If the rope is perpendicular to the ladder and the ladder makes an angle 60° with the horizontal, find the tension of the rope and the reactions at the ends.

Find also the tension when a boy weighing 100 lbs. is at the top of the ladder.

15. A rod of weight 10 lbs. rests with its ends on a smooth horizontal plane and on a smooth plane of inclination 60° . It is kept at rest at an angle 30° with the horizontal, by a horizontal string attached to the lower end. Find the tension of the string and the reactions at the ends of the rod.

16. A rod is in limiting equilibrium, resting horizontally with its ends on two inclined planes at right angles, one of which makes an angle α ($< 45^\circ$) with the horizontal. If the coefficient of friction μ is the same at both ends, prove that

$$\mu = \frac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha},$$

and that

$$\lambda = 45^\circ - \alpha.$$

17. If in Question 16 the steeper plane is smooth and the rod rests as before, prove that

$$\begin{aligned}\mu &= \cot 2\alpha, \\ \lambda &= 90^\circ - 2\alpha.\end{aligned}$$

18. A weightless rod of length l is placed between two rough vertical posts at a distance d apart less than l . If μ is the coefficient of friction, what is the lowest point of the rod at which a weight can be hung without the rod slipping down?

19. An equilateral triangle is formed by three uniform rods rigidly attached to one another at their ends, and is suspended on a rough horizontal peg, the angle of friction between peg and rods being λ . Find the range of possible equilibrium positions, excluding those in which the peg is at a vertex of the triangle.

20. A cotton reel, partly emptied, rests on a rough plane inclined at an angle α to the horizontal. The end of the thread is held so that it comes off parallel to the plane. If the radius of the reel is a and the radius of the cylinder formed by the cotton is b , find the least coefficient of friction necessary to prevent it from sliding down.

21. A circular disk of weight W , whose centre is C , is weighted at one point A on the circumference by a weight W' . It rests in equilibrium in a vertical plane on a rough plane inclined at an angle α to the horizontal. Find the angle CA makes with the horizontal, and find the least coefficient of friction in order that it may rest in this way.

Examine the case when $W = W'$, $\alpha = 30^\circ$.

22. A uniform gate, 5 ft. high by 6 ft., weighs 150 lbs., and is supported by two hinges placed respectively 6 ins. from the top and bottom of the gate. If all the weight is borne by the lower hinge, find the reaction at each hinge.

23. A square plate $ABCD$, of weight W , rests in a vertical plane, being supported by a hinge at A and resting against a smooth peg at B . If AB makes an angle θ with the vertical, find the reaction at A , and the pressure on the peg.

24. A rod AB , of length b , is supported in a horizontal position by a string attached to the end B , while A rests against a rough vertical wall. If the string is attached to the wall at a height h above the rod, shew that the coefficient of friction must be at least as great as h/b .

25. A pole, of weight 300 lbs., is being raised by a rope attached to one end, the other end of the pole being on the ground. If the pole slips when it makes an angle 30° with the horizontal and the rope is perpendicular to the pole, find the coefficient of friction and the tension of the rope at the time.

71. Other Methods of Reduction of a System of Forces.

Forces may be reduced to simpler forms in other ways than that of Article 65, but they are of less general importance.

For example, any system of forces in a plane can be reduced to two forces acting through two given points A and B .

For it is only necessary to take a point C on the line of action of P and join CA , CB , and the force can be replaced by components along CA and CB . Hence any number of forces can be reduced to one set through A , and another set through B . These can be compounded into two forces through A and B respectively.

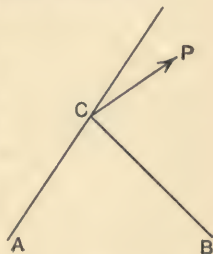


FIG. 107.

Again, any forces may be reduced to three forces acting along the sides of a given triangle ABC.

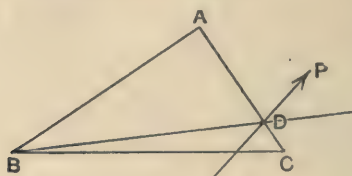


FIG. 108.

Any force P must cut two at least of the sides (or sides produced) of the triangle. Let it cut AC in D . Join BD . P can be resolved into components along DB and DC . The force along DB can further be resolved into components along BA and BC . Thus P is replaced by three forces along AB , BC , CA , and so for any number of forces.

Example. Prove that in this case the necessary and sufficient conditions for equilibrium are that the total forces along AB , BC , CA should separately vanish.

72. Algebraical (or Analytical) Methods of Compounding Forces.

Suppose the forces act at points $A_1, A_2, A_3, \dots$, whose positions are given by their coordinates relative to axes Ox and Oy , and that the magnitudes of the forces are $P_1, P_2, P_3, \dots$ and their directions make angles $\alpha_1, \alpha_2, \alpha_3, \dots$ with the axis Ox .

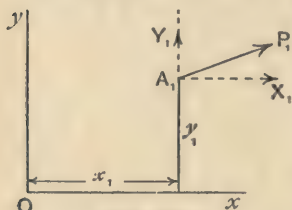


FIG. 109.

First resolve P_1 into components X_1 and Y_1 parallel to Ox and Oy .

Now Y_1 can be replaced by

$$Y_1 \text{ at } O \text{ and a couple } Y_1 x_1.$$

X_1 can be replaced by

X_1 at O and a couple $-X_1y_1$.

Hence P_1 is replaced by

X_1 along Ox ,

Y_1 along Oy ,

and a couple $Y_1x_1 - X_1y_1$.

Hence, adding up for all the forces, we have

ΣX along Ox ,

ΣY along Oy ,

and a couple $\Sigma(Yx - Xy)$.

If $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma(Yx - Xy) = 0$, we have equilibrium.

If $\Sigma X = 0$ and $\Sigma Y = 0$, the forces reduce to a couple.

In other cases they reduce to a single force.

We can find the equation of the straight line along which the resultant force (when there is one) acts as follows.

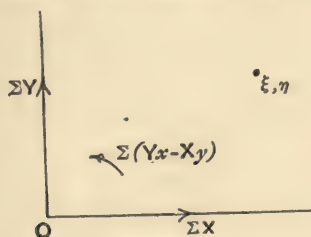


FIG. 110.

The total moment about any point whose coordinates are ξ, η , is

$$\eta \Sigma X - \xi \Sigma Y + \Sigma(Yx - Xy).$$

If ξ, η is on the resultant force of the system, this moment must be zero, or $\eta \Sigma X - \xi \Sigma Y + \Sigma(Yx - Xy) = 0$.

This equation gives a relation between ξ and η , and is therefore the equation of the straight line along which the resultant acts.

EXAMPLES.

1. A weightless rod CD is hinged at C and carries a weight W at D. The rod is tied back by a string AB attached to a point A vertically above C. Prove that the tension of the string is

$$Wl \sin(A+B)/b \sin A,$$

if l is the length of the rod, and find the components of the reaction at C.

2. A rod AB, of length $2l$ and weight W , presses against a smooth vertical wall at A, passes over a smooth peg at a distance h from the wall, and is held by a string attached at B and perpendicular to the rod. If the rod makes an angle α with the upward vertical, find the pressures at the wall and the peg, and the tension of the string.

3. A rod AC, of weight W , can rotate in a vertical plane about a horizontal axis at B, which divides the rod into portions of lengths a and b . The end A abuts on a smooth vertical wall at a distance h from B. If a downward force P is exerted at C, find the pressure at A and the reaction at B.

4. A bracket ABC consists of a uniform rod bent at right angles at B and of weight w per unit length. It is screwed to a vertical wall at D in AB, and B is above A. Supposing the bracket to touch the wall at A only, but AB to be practically vertical, and a weight W to be placed on BC at E, find the tension of the screw and the pressure on the wall. Neglect friction, and take $AB=a$, $BC=b$, $AD=c$, $BE=d$.

5. A cubical block of stone, whose edges are $2a$ ft. and weight W lbs., lies on a horizontal plane and is to be turned over by a horizontal force applied at the middle of one of the upper edges and perpendicular to that edge. Find the least coefficient of friction that this may be done without the block slipping, and find the force P exerted when the block has been turned through any angle θ .

6. A cubical block of edge $2a$ standing on a rough plane (the coefficient of friction being μ) is pulled by a force P in a vertical plane bisecting four edges, and P makes a fixed angle θ above the horizontal. P being gradually increased, the block at last slides without overturning. Shew that the line of action of P must intersect the base within a distance $a/\mu \tan \theta$ from the centre of the base.

7. A solid circular cylinder of weight W , radius r , and length $2l$, standing on a rough horizontal plane, is tilted by a force P applied at the bottom edge at right angles to the base. Assuming the cylinder not to slip, shew that when the inclination of the base to the horizontal is θ ,

$$P = W(r \cos \theta - l \sin \theta)/2r.$$

8. A sphere is resting on a rough horizontal plane. A pressure P is applied at a point the radius to which makes an angle α with the vertical. If equilibrium still subsists, find the direction of P , and shew that P cannot be greater than

$$\mu W / \left(\sin \frac{\alpha}{2} - \mu \cos \frac{\alpha}{2} \right),$$

if $\tan \frac{\alpha}{2}$ is greater than μ .

9. A rectangular board $ABCD$, of weight W , rests with its plane vertical and the side AB below DC , these sides making an angle α with the horizontal, and is supported by a rough peg at the middle point of AB and with the side CD pressing against a smooth peg P . Find the reactions at the pegs, and if the angle of friction (λ) is less than α , prove that the distance of P from the middle point of CD cannot be greater than $BC \sin \alpha \sin \lambda / 2 \sin (\alpha - \lambda)$.

10. A square plate of edge $2a$ stands in a vertical plane with two sides horizontal, the lower of which rests on a rough peg at a distance d from a rough vertical wall against which an upper corner of the plate presses, the vertical side through this corner being otherwise just clear of the wall. Shew that if the coefficients of friction are

the same at the two contacts, they cannot be less than $\sqrt{\frac{d}{a}} - 1$.

11. A rod AB rests with one end on a rough horizontal plane, and passes over a smooth peg at a height h above the plane. Shew that if the rod makes an angle θ with the horizontal, the least coefficient of friction is $l \sin^2 \theta \cos \theta / (h - l \sin \theta \cos^2 \theta)$ if the length of the rod is $2l$.

12. A drawer, of breadth $2a$ and depth (from front to back) $2b$, has two symmetrically placed handles $2d$ apart. Shew that (neglecting the weight of the drawer) the drawer cannot be pulled out by one handle if the coefficient of friction between the drawer and its guides is greater than b/d .

13. A rectangular window frame, 3 ft. wide and 2 ft. 6 ins. high, and of weight W , is counterpoised by weights, each $\frac{W}{2}$, attached to cords passing over pulleys and attached to the upper corners of the frame. What is the least coefficient of friction necessary that if one cord breaks the frame may not fall?

14. A vertical cylindrical bar, of weight W , can slide in two fixed horizontal rings A , B , of slightly greater diameter, at a distance d apart. The bar is to be raised slowly by a vertical force P at a

distance c from the axis of the cylinder. Assuming a coefficient of friction μ at the contact with each ring, prove that

$$P = W / \left(1 - \frac{2\mu c}{d} \right).$$

Explain what happens if $2\mu c > d$.

15. A ladder of length $2l$ is in contact with a vertical wall and a horizontal floor, the angle of friction being λ at each contact. If the weight of the ladder acts at a distance kl from the middle point downwards, prove that the limiting inclination θ to the vertical is

$$\cot^{-1}(\cot 2\lambda - k \operatorname{cosec} 2\lambda).$$

If the coefficient of friction is $\frac{1}{3}$, prove that the ladder will slip at any angle if the centre of mass is more than one-tenth of the length from the lower end.

16. A gasalier with two straight arms, each of length a in a line at right angles to the descending tube of length l , is hanging out of plumb from the joint at the top. A horizontal force P , applied at the lower end of the descending tube, brings the gasalier plumb. Find the weight at the end of the arm that would do so.

17. A plate can turn in its plane, which is horizontal about an axis O . A cord passes round two smooth pegs in the same line as O , and the ends are attached to two fixed points A and B outside the plate and in the same plane as it. The strings run from A and B perpendicular to AB . $AB = a$, $BO = b$. A couple L is applied to the plate so as to strain the string. Find its tension.

18. A ladder weighing 200 lbs. rests at an angle 65° with the horizontal. If the friction at the lower end is 40 lbs.-wt., find the normal reaction there and the friction and normal reaction at the upper end.

19. A rectangular box is tilted up so that the two long bottom edges rest one against a smooth vertical wall, and the other on a rough horizontal floor, and the bottom makes an angle α with the vertical. If the breadth and height are $2b$ and $2h$ respectively, shew that it is on the point of slipping if $\tan \alpha = 2\mu + \frac{h}{b}$.

CHAPTER VI.

CONNECTED BODIES.

73. In very many questions we have to deal with two or more bodies pressing against one another or attached to one another in some way, and it is often necessary then to consider the equilibrium of each. The reactions between the bodies have to be marked in accordance with the law of Action and Reaction. that is to say, the reactions marked on the two must be equal in magnitude but in opposite directions.

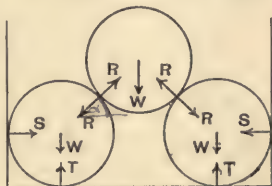


FIG. 111.

Take a simple example. Suppose three similar cylinders placed one on top of the other two in a box with horizontal base and vertical sides, whose distance apart is greater than twice the diameter of the cylinders. The forces are as in the figure, the upper cylinder pressing on each of the lower ones with a force R , and the lower ones pressing on the upper with the same force. The sides of the box press against the lower ones with force S , and the base presses on the cylinders with force T .

If the plane through the axis of the upper and a lower cylinder makes an angle α with the horizontal, we have :

(i) For the upper cylinder,

$$\text{vertically, } W = 2R \sin \alpha. \dots\dots\dots(1)$$

(ii) For a lower cylinder,

$$\text{vertically, } W + R \sin \alpha = T; \dots\dots\dots(2)$$

$$\text{horizontally, } S = R \cos \alpha. \dots\dots\dots(3)$$

As the forces on a cylinder all pass through one point, there is nothing additional gained by taking moments. In fact, these three equations give the three unknowns R , S , T .

$$R = \frac{W}{2 \sin \alpha},$$

$$T = \frac{3W}{2},$$

$$S = \frac{W}{2} \cot \alpha.$$

When particles are connected by a string, it follows, by the reasoning of Chapter I., that the tension exerted by the string on each body is the same, provided the weight of the string is negligible, and there is no friction between the string and any surface with which it is in contact.

74. Jointed Rods.

A very important case of connected bodies is that of two

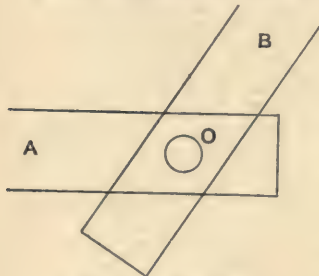


FIG. 112.—Pin joint.

rods jointed together by a pin run through the two. A good illustration is a pair of compasses or simply two laths nailed together by a single nail. Such a joint is called a **pin joint**. We have here three bodies to consider, the rods, which we will call A and B , and the pin C , the centre of the pin being at O .

The forces between these may be expressed as actions :

On A , due to B and C .

„ B , „ C „ A .

„ C „ A „ B .

In each case these reactions may be represented by a single force through O and a couple.

In the present chapter we will deal with the case when we have other forces acting on the rods, such as their weights, but the pin can be regarded as weightless, and the only forces acting on it are the reactions due to the rods. In this case it is not usually necessary to consider the pin separately at all ; in other words, the pin may be considered as a part of one of the rods, and then we have only to consider the reactions between the two rods.

75. If the pair of rods is placed on a horizontal plane (so that we do not have to consider the effect of their weight), and we try to alter the angle between the two by applying forces to them (as in opening or shutting a pair of compasses), it will be found that a resistance seems to be exerted to the motion. This is due to friction at the joint.

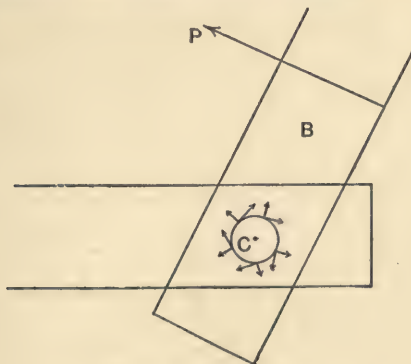


FIG. 113.—Rough joint.

Thus, if, for simplicity, we think of A and the pin C as fixed, and A and B not directly in contact, and try to turn B round by a force P, there will be pressure between B and C such that at each point of the circumference of C there is a normal reaction and friction. It will be seen at once that if there were no friction, the resultant reaction would pass through the centre

of the pin, and the slightest force P would turn B round. In such a case the rods are said to be **smoothly jointed**. As has been remarked before in other cases involving friction, when rods are said to be smoothly jointed, it will not necessarily be understood that no friction *could* be brought into play at the joint, but simply that friction *is not* brought into play in the particular position of the rods. When friction does act, and the reactions are reduced to a force at O and a couple, the moment of the couple (by Fig. 113) is the sum of the moments of the frictional forces, and consequently the couple is called the **Friction Couple**.

If we place A and B in a vertical plane, thinking of A as fixed as before, then, if the rods are smoothly jointed, B will

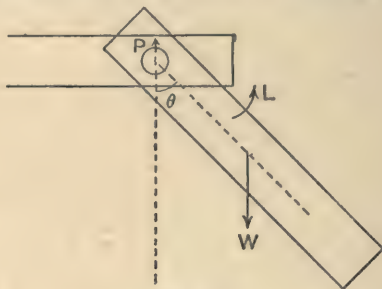


FIG. 114.

hang vertically, but if friction can be brought into play B may rest inclined to the vertical.

Thinking of the reaction as denoted by a force P at O and a couple L , the only other force being W , and the length of the rod $2a$, and inclination to the vertical θ , then we see that the force P must be vertical and equal to W , and

$$L = Wa \sin \theta.$$

Also the maximum friction couple can be found by increasing θ till the rod will not remain at rest, or if it will rest even when placed horizontally, extra weights can be placed on it,

and the total moment, when it just begins to move, is equal to the maximum friction couple.

In the next chapter we will deal with cases where the reactions may be considered as acting only between rods and pins and not between the two rods.

76. In solving examples it is best to draw the rods separated from one another, and mark the reactions on each. As the direction of the reaction will not usually be known at first it is best to mark two resolutes (generally horizontal and vertical) of each reaction, drawing them in the directions which will least confuse the figures. If any force comes out negative it will only mean that it acts in the opposite direction to that marked on the diagram.

Examples. 1. Two equal rods AC, BC, each of weight 10 lbs., are smoothly jointed at C and supported at points A, B, in the same horizontal line. If the rods each make an angle 30° with the vertical, find the forces at the supports and their action at C.

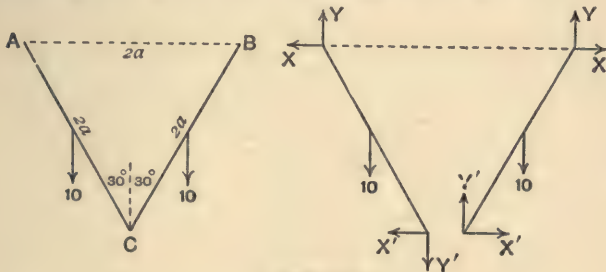


FIG. 115.

The reactions at A and B will be symmetrical; at C put in the two components of the reaction as shewn. The forces thus act on the body to which the ends of the arrows are attached. (There is no friction couple at C, as the rods are smoothly jointed.)

Take each rod separately.

For AC, vertically, $Y = 10 + Y' ; \dots\dots\dots(1)$

horizontally, $X + X' = 0 ; \dots\dots\dots(2)$

moments about C,

$Y \cdot 2a \sin 30^\circ = X \cdot 2a \cos 30^\circ + 10a \sin 30^\circ. \dots\dots\dots(3)$

For BC, vertically, $Y + Y' = 10$(4)

The other equations for BC are the same as for AC.

Hence we have four equations for four unknowns.

From (1) and (4) we get

$$Y' = 0, \quad Y = 10,$$

and (3) then gives $X \cdot 2a \frac{\sqrt{3}}{2} = 10a \times \frac{1}{2},$

$$X = 5/\sqrt{3} = 2.89 \text{ lbs.-wt.},$$

$$X' = -2.89 \text{ lbs.-wt.}$$

We can now write down the resultant force at A as we have the two components. Its magnitude is $\sqrt{\frac{25}{3} + 100} = 10.41$, and its direction makes an angle with the vertical equal to $\tan^{-1} 2\sqrt{3}$ or $73^\circ 54'$.

The reaction at C is horizontal and acts in the opposite direction to that drawn in the figure; in other words, the rod AC pulls at BC, and BC pulls at AC, as is obvious must be the case.

77. Frequently considerations of **symmetry** shorten the work. In the present example we used it in marking the forces at A and B as the same. But if we look further into the matter we will find that we could have said from the first that the reaction at C is horizontal. For Newton's Law of Action and Reaction requires that the vertical components on the two rods should be in opposite directions, as shewn in the figure, but symmetry requires that these components should be in the same direction; these two requirements can only be satisfied by the vertical components being zero. We can therefore leave out Y' in the diagram, and we have only three unknowns and only have to use the equations for one rod.

It is always possible to write down three equations for each rod. We ought then to have the correct number of equations to find the unknowns, and can solve algebraically. But this process may be awkward, and it is frequently possible to form an equation containing one of the unknowns only, and so find it and continue with other equations, selecting them so that each new equation contains one new unknown. In doing this it is frequently advantageous to form equations for two or

more rods treated as one body, the reactions between these rods then not coming into the question.

2. Thus, take the example above, modified only by making one rod 20 lbs.-wt., and the other 10.

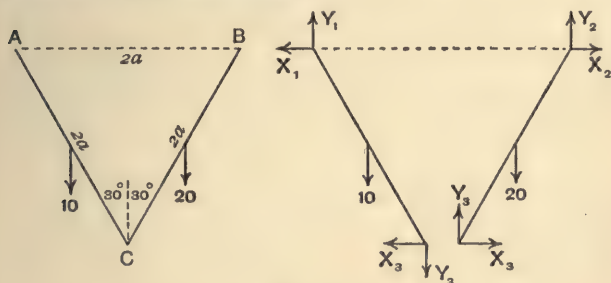


FIG. 116.

There is now no symmetry, and therefore the six resolutes of the reactions must all be marked differently.

We can proceed conveniently as follows :

Take moments about A for the two rods together, then X_3, Y_3 do not come into the equation ; X_1, Y_1, X_2 have no moments about A ;

$$\therefore Y_2 \cdot 2a = 10 \times \frac{a}{2} + 20 \times \frac{3a}{2} = 35a,$$

$$Y_2 = \frac{35}{2};$$

vertically for the two,

$$Y_1 + Y_2 = 10 + 20,$$

$$Y_1 = \frac{25}{2};$$

vertically for AB,

$$Y_3 + 10 = Y_1 = \frac{25}{2};$$

$$\therefore Y_3 = \frac{5}{2};$$

moments about C for AC,

$$X_1 \cdot 2a \cos 30^\circ + 10a \sin 30^\circ = Y_1 \cdot 2a \sin 30^\circ;$$

$$\therefore X_1 \sqrt{3} + 5 = \frac{25}{2},$$

$$X_1 = \frac{15}{2\sqrt{3}} = \frac{5}{2}\sqrt{3} = 4.33 \text{ lbs.-wt.};$$

horizontally for AC,

$$X_1 + X_3 = 0,$$

$$X_3 = -\frac{5}{2}\sqrt{3} = -4.33;$$

horizontally for BC,

$$X_3 = -X_3 + \frac{5}{2}\sqrt{3} = 4.33.$$

3. Two equal uniform rods are jointed at one end of each. The joint is not smooth, and if one is kept fixed the other can just be raised to an angle of 45° in the vertical plane without its moving when released. The rods are placed in a vertical plane with their ends resting on a rough table, the coefficient at the table being 0.5. If the same couple is always required to move the joint, find the greatest angle to which they can be separated without slipping.

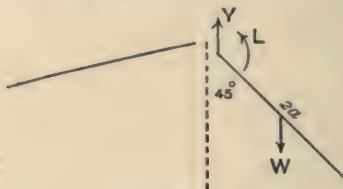


FIG. 117.

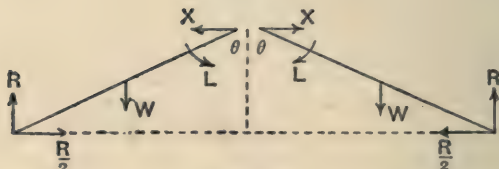


FIG. 118.

Fig. 117 represents the given data. Fig. 118 represents the limiting position in the problem. In Fig. 118 the forces are symmetrical, and we will only have to form the equations for one rod.

From Fig. 117, $Wa \sin 45^\circ = L$.

From Fig. 118, vertically, $R = W$;

horizontally $\frac{R}{2} = X$;

moments about C, $X \cdot 2a \cos \theta - Wa \sin \theta + L = 0$;

$$\therefore \frac{W}{2} \cdot 2a \cos \theta - Wa \sin \theta + Wa \sin 45^\circ = 0;$$

$$\therefore \cos \theta - \sin \theta = -\sin 45^\circ = -\frac{1}{\sqrt{2}};$$

$$\begin{aligned}\therefore \cos \theta \frac{1}{\sqrt{2}} - \sin \theta \frac{1}{\sqrt{2}} &= -\frac{1}{2}; \\ \therefore \cos(\theta + 45^\circ) &= \cos 120^\circ; \\ \therefore \theta + 45^\circ &= 120^\circ; \\ \therefore \theta &= 75^\circ.\end{aligned}$$

4. Two rods, AB, BC, are jointed at B, and attached to fixed points A, C in the same vertical, A being above C. If BC is horizontal, and AB makes an angle 30° with the vertical, and the weights are 20 lbs. and 10 lbs. respectively, find the reactions at A, B, and C.

5. Solve the same question as No. 4, except that BC makes an angle 60° with the vertical.

6. AB, BC are two rods jointed at B, and of lengths 3 ft. and 4 ft., and weights 15 and 20 lbs., supported at two points A and C in the same horizontal line. If AC = 5 ft., find the reactions at A, B, and C.

7. Solve the same question as No. 6, except that AC is vertical with A above B.

8. Three rods, AB, BC, CD, of equal weight and equal length $2a$, are smoothly hinged at B and C and supported at points A and D in the same horizontal line. If AB and CD make angles α with the horizontal, find the reactions at A and B.

9. Three equal rods, as in Question 8, are supported at points A and D not in the same horizontal. If α, β, γ are the angles the rods make with the horizontal, prove that $\tan \alpha, \tan \beta, \tan \gamma$ are in A.P., and find the components of the reactions at A, B, C, D.

10. Three rods, AB, BC, CD, of lengths $2a, 2b, 2a$, but of equal weight, smoothly hinged at B and C, rest symmetrically in a vertical plane with AB, CD, on two smooth pegs E, F, in the same horizontal. If AE = 3EB, prove that AB, CD must make angles 30° with the horizontal, and so also does the reaction at B. Find the reactions.

11. Three rods, AB, BC, CD, are of equal weight W , and AB, CD are of equal length. They rest, as in Question 10, over two pegs E, F at a distance $2d$ apart in the same horizontal. If BC is of length $2a$, and AB makes an angle θ with the horizontal, prove that the length of AB is $3(d - b) \sec^3 \theta$, and that the reaction at B makes an angle $\tan^{-1}(\frac{1}{3} \cot \theta)$ with the horizontal, and find the reactions at B and E.

12. A rod AB, of weight w , is hinged at A and carries a weight W at B. It is kept at an inclined position by a weightless rod CD jointed to it at a point C, and to a fixed point D vertically below A. Shew that the thrust in CD is

$$(2W + w)AB \cdot CD / 2AC \cdot AD,$$

and find the reaction at A.

13. Two rods, AB, BC, of equal length, but of weights 20 and 30 lbs., are smoothly hinged together at B and connected by a string AC. They stand in a vertical plane with A, C, on a smooth horizontal plane. If ABC is an equilateral triangle, find the tension of the string and the reaction at B.

14. AB is a rod 20 ft. long and weighing 40 lbs., capable of turning in a vertical plane about a smooth hinge at A. A smooth weightless ring D slides on the rod and is attached by a string 5 ft. long to a point C, 13 ft. vertically above A. Shew that in the position of equilibrium the string is perpendicular to the rod, and find its tension and the reaction at A.

15. If a rod supported as in Question 14 has a weight W and length $2l$, and the ring, instead of being weightless, has a weight w , but is smooth, and if $AC = b$, $CD = a$, obtain equations to determine the position of equilibrium.

16. Two rods, AC, CB, of the same length, but of weights W , $2W$, are smoothly jointed at C, and stand in a vertical plane with their ends on a rough horizontal plane. The coefficient of friction μ being the same at A and B, shew that the greatest angle the rods can make with one another is 2θ , where $\tan \theta = 5\mu/3$.

In what way do the rods begin to move when limiting equilibrium is passed?

17. AC, CB are two rods smoothly jointed at C and fixed to points A, B in the same vertical. If the weights of the rods are in the ratio $\cot A : \cot B$, shew that the reaction at C is horizontal and equal to $W \sin A \sin B / 2 \sin C$, where W is the total weight of the rods.

18. A square board ABCD, of weight W , rests with one edge AB on the ground, and is supported by a weightless rod EF hinged to the board at a point E, and resting on the ground at F, EF being in the vertical plane bisecting AB. If $AB = a$ and E is at a distance b from AB, find the thrust in the rod and the pressure of the square on the ground when the square makes an angle α with the ground, and EF an angle θ . Find also the least coefficient of friction, supposed the same at each contact, for it to remain in this position.

19. A cubical block of weight W rests with one edge AB on a smooth horizontal plane, and the base maintained at an angle 2α with the plane by a smooth roller of radius r between it, and the plane with its axis parallel to AB. Equilibrium is maintained by two horizontal forces P, Q, one on the edge AB, and the other on the axis of the roller. Prove that

$$P = Q = 2\sqrt{2} \frac{Wa}{r} \cos\left(\frac{\pi}{4} + 2\alpha\right) \sin^2 \alpha,$$

where $AB = 2a$.

20. Two rods, AB, AC, are of the same material and thickness, but AC is double the length of AB and of weight W lbs. They are smoothly jointed at A and supported at points B and C, so that they each make an angle α with the vertical. Find the reactions at A, B, C.

21. Three rods, AB, BC, CD, each of length $2a$, are jointed together in such a way that each joint offers a maximum couple L of resistance to turning. AB being held fixed, find the greatest horizontal distance of D from B.

22. Two rods, AB, AC, smoothly jointed at A, rest in a vertical plane with their ends on a smooth horizontal plane. A horizontal string joins two points D, E on the rods. Prove that its tension is

$$\frac{1}{4}W \cdot AB \sin \theta / AP,$$

where θ is the angle the rods make with the horizontal, W is the total weight of the rods, and P is the middle point of the string.

23. Shew that the force required to support a cylinder of weight W on a plane inclined at an angle α to the horizontal by a crowbar of length l inclined at an angle β to the horizontal is

$$Wr \sin \alpha \tan \frac{1}{2}(\alpha + \beta) / l \sin (\alpha + \beta).$$

24. Three similar rods AB, BC, CD, each of length $2a$, rest symmetrically on two smooth pegs E, F, in the same horizontal line, and under the outer rods. Find the distance between the pegs

(i) When the three rods are horizontal.

(ii) When AB, CD are inclined at an angle 30° to the horizontal, BC being horizontal.

25. Two equal rods, AB, CD, each of weight W , are hinged together at their middle points E and stand in a vertical plane on a smooth horizontal table, being kept from collapsing by a string joining their lower ends A, C. If the rods make angles α with the horizontal, find the tension of the string.

26. Work out the same question as No. 25, but with the string joining A to the middle point of CE. Also find the reaction at the hinge.

27. Two equal heavy circular cylinders are placed symmetrically in the angle formed by two rough planes equally inclined to the vertical. A rough wedge is pressed down symmetrically between the two. If 2α is the angle between the planes, and λ the angle of friction at all contacts, and 2β the angle of the wedge, shew that the wedge cannot force up the cylinders unless $\alpha - \beta > 2\lambda$.

If this condition is satisfied, find the least pressure on the wedge which will force up the cylinders, and prove that they begin to move by slipping on the wedge.

28. A rough hemisphere rests with its base on a rough horizontal plane. The lower end (rough) of a stick sliding in a smooth vertical tube rests on the hemisphere. Shew that the hemisphere begins to slide when the inclination α to the vertical of the radius to the lower end of the stick is given by

$$W' \tan (\alpha - \lambda) = (W + W') \tan \lambda,$$

where λ is the angle of friction at both contacts.

29. A heavy framework ABC, formed of bars of weight w per unit length, is suspended from C, and weights W_1 , W_2 are suspended from A and B respectively, so that AB is horizontal. Shew that

$$\{2W_1 + w(b+c)\}b \cos A = \{2W_2 + w(a+c)\}a \cos B,$$

and find the component reactions at A.

30. AC, BC are two equal rods, each of weight w , smoothly jointed at C, and jointed to points A and B in a vertical line so that ABC is an equilateral triangle. A weight W is hung at C; find the reactions at A, B, and C.

CHAPTER VII.

WEIGHTLESS FRAMEWORKS.

78. Frameworks such as girders, which carry heavy loads, are treated rather differently from the rods in the last chapter. For the bars of the girder can be treated in a first approximation at any rate, as weightless, that is to say, their weights are small compared with the loads carried.

Any load borne by a bar may, moreover, be replaced by two loads at the two ends of the bar, and may be supposed to act on the pins and not on the bar. Also, the bars may be supposed to act and react on the pins only, and not on one another. Lastly, in a frame at rest and undeformable, friction at the joints is practically non-existent, and may be neglected altogether.

Thinking of the ideal case, when a single pin passes through the ends of the bars, it is necessary, of course, that there should be sufficient bars to keep the frame from deforming; in other words, to make it rigid, and this requirement will almost always be followed in any actual frame, though examples of roof trusses not satisfying the condition are sometimes found. These remain undeformed for symmetrical loads, and for other loads rigidity is maintained by the stiffness of the joints (say by extra bolts there).

We will shew immediately (Art. 81) that if a framework has n joints, it requires $2n - 3$ bars to make it rigid.

As an example, a quadrilateral framework ABCD of any shape is not rigid; it can be deformed. But it can be made rigid by putting a bar across from A to C, when it can no longer be deformed. We notice that here there will be 4 joints and

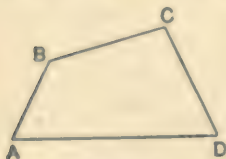


FIG. 119.

$$2 \times 4 - 3 = 5 \text{ bars.}$$

If we put another bar across from B to D, we have more bars than necessary for rigidity, and it is found that the stresses cannot then be calculated by the ordinary methods of statics used so far, and they are, in fact, indeterminate, and the frame may have stresses in it without external force acting on it at all. Frames such as this, with more than the necessary number of bars, are called **Redundant**, and will not be dealt with in this book.

79. Taking, then, the usual rigid framework, any bar is regarded as weightless and acted on only by the reactions due to the pins at each end. In other words, the bar is acted on by two forces only, one at each end, and as the bar is in equilibrium, these two forces must be equal, opposite, and in the same straight line, that is to say, along the length of the bar. Thus the bar is said to be in a state of stress, and the force at either end is called the **Stress** in the bar, and may be a **Tension** or **Thrust**. Each section of the bar will have the same tension or thrust across it, exactly as in the case of the tension of a string (Art. 7). If the rod is pulling at each pin, and consequently the pins are pulling at it, it is in a state of tension; if it is being pushed by the two pins, it is in a state of thrust.

One can frequently tell at sight, with practice, whether a given bar of a framework will be in tension or thrust, but I consider it best to mark the forces in a uniform way, and the positive and negative signs in the results will tell definitely whether the stress is a tension or thrust.

Let us take the simplest possible case—a framework forming an equilateral triangle with its base resting on two supports at its ends, and supporting a weight of, say, 100 lbs. at its vertex.

Fig. 120 shews the framework as a whole.

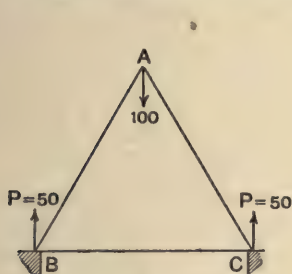


FIG. 120.

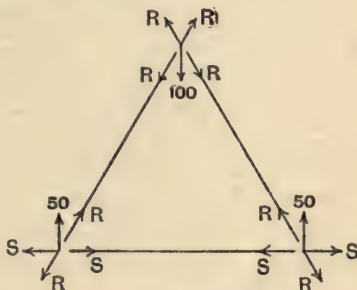


FIG. 121.

Fig. 121 shews it with the rods separated to mark the forces clearly.

The forces are marked in the most convenient way for drawing, and it will be seen that if a stress thus marked is positive (that is, acts in the direction shewn) the pins are pushing at the rod or the rod is in a state of thrust. If the stress is negative (or in the opposite direction to that marked) the pins are pulling at the rod, and the rod is in tension.

Symmetry has been taken into account in marking the forces.

To find the forces we now have only to use the equations for the pins.

For pin A, vertically,

$$2R \cos 30^\circ = 100,$$

$$R = 100/\sqrt{3} = 57.7 \text{ lbs.-wt.};$$

for B, horizontally,

$$S + R \cos 60^\circ = 0,$$

$$S = -28.9 \text{ lbs.-wt.}$$

Consequently the inclined rods are in thrust, the horizontal one in tension.

Usually the external forces—the pressures on the supports—are found first by forming the equations for the framework taken as a whole, thus avoiding the introduction of any internal reactions.

When the external forces have been found the work of dealing with the equations for the pins should always start (if possible) at a joint where there are only two unknown reactions, and should always continue in the same way joint by joint, always taking points where there are not more than two unknown reactions, for we can then write down the two equations for the pin and find the two unknowns. If this cannot be done, the work is much complicated by the necessity of writing down a number of equations and then solving.

80. Take a little more complicated example, as represented in Fig. 122.

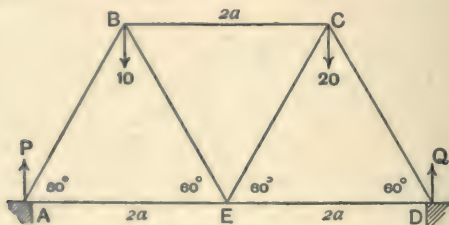


FIG. 122.

As before, Fig. 122 shews the frame as a whole, with the external forces only. Fig. 123 shews it with rods and pins separated, and all the reactions marked.

First take the frame as a whole to find the external forces P and Q .

Moments about A ,

$$Q \times 4a = 10 \times a + 20 \times 3a, \quad Q = 17.5;$$

vertically for the whole frame,

$$P + Q = 30,$$

$$\therefore P = 12.5.$$

For joint A, vertically,

$$R \sin 60^\circ = P,$$

$$\therefore R = 25/\sqrt{3} = 14.44;$$

horizontally,

$$W + R \cos 60^\circ = 0,$$

$$\therefore W = -25/2\sqrt{3} = -7.22.$$

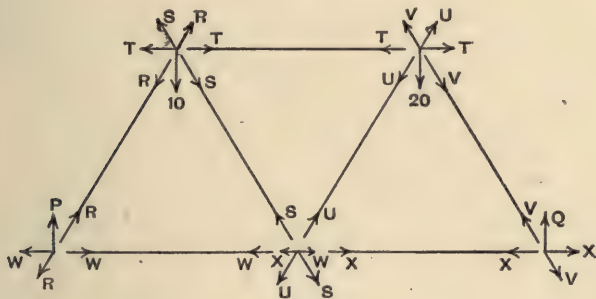


FIG. 123.

For B, vertically,

$$R \sin 60^\circ + S \sin 60^\circ = 10,$$

$$\therefore S = \frac{20}{\sqrt{3}} - \frac{25}{\sqrt{3}} = -\frac{5}{\sqrt{3}} = -2.89;$$

horizontally,

$$T + S \cos 60^\circ = R \cos 60^\circ,$$

$$\therefore T = \frac{5}{2\sqrt{3}} + \frac{25}{2\sqrt{3}} = \frac{15}{\sqrt{3}} = 5\sqrt{3} = 8.66.$$

For E, vertically,

$$U \sin 60^\circ + S \sin 60^\circ = 0,$$

$$\therefore U = \frac{5}{\sqrt{3}} = 2.89;$$

horizontally,

$$X + U \cos 60^\circ = W + S \cos 60^\circ,$$

$$\therefore X = -\frac{5}{2\sqrt{3}} - \frac{25}{2\sqrt{3}} - \frac{5}{2\sqrt{3}} = -\frac{35}{2\sqrt{3}} = -10.10.$$

At D, vertically,

$$V \sin 60^\circ = Q,$$

$$\therefore V = \frac{35}{\sqrt{3}} = 20.21.$$

This completes the determination of the stresses. The horizontal equation for D and the two equations for C may now be used as checks.

Instead of drawing the whole Fig. 123, it is sufficient to draw the pins alone, and mark the rods in Fig. 122 with the stresses, thus :

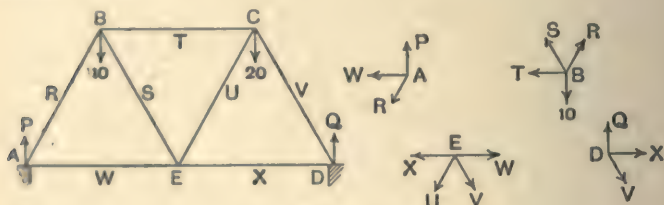


FIG. 124.

81. With regard to the number of joints and rods in a just rigid frame, it will be seen that if we start to build up a frame, gradually starting from a triangle, each extra joint requires two extra rods to fix it in position. Thus, in the figure, starting with ABE, we add two rods BC and EC, fixing the joint C, and so on. Now, in the triangle there are 3 joints and 3 rods; hence if we add $n - 3$ more joints, and consequently $2n - 6$ more bars, we get a frame with n joints and $2n - 3$ bars.

In finding the stresses we get $2n$ equations from the n joints, and these are the correct number to give the $2n - 3$ stresses in the bars, and leave 3 equations which determine the external forces, or, in other words, are required for the equilibrium of the frame as a whole.

Thus, in the example worked out above, there are 5 joints and $2 \times 5 - 3 = 7$ bars. We used the equations for the frame

as a whole at the start, with the result that three of the equations for the joints (one for D and two for C) were not wanted, and could be used as a check on the arithmetic.

82. Heavy Bars.

While we do not intend to work out frameworks where the weights of the bars are to be taken into account, a few remarks on the method of procedure should be made in connection with the designing of a structure for a given purpose.

When the weights of the bars are to be taken into account the reactions are no longer along the lengths of the bars. But we may calculate the stresses by first replacing the weight of each bar by half its weight at each end, and suppose this to act on the pins as before. We may now calculate the stresses as for a weightless framework, and it will be found that the true reaction exerted by a bar on a joint is got by compounding the reaction thus found with half the weight of the bar, and the reaction on the bar is got by combining the reaction found by the weightless framework method with a force equal and opposite to half the weight of the bar. This method will be most useful in graphical work.

That this is a correct method will be seen by examining a simple case such as the framework of Art. 79.

Suppose the bars, instead of being weightless, have a weight of 20 lbs. each. Shifting these weights to the pins we have 120 lbs. at A, and 80 upwards and 20 downwards at B and C. The force R' must now be found. But it is obvious that if R' on the pin is compounded with a force of 10 downwards to give a force R'' , the pin will be in equilibrium under the two forces R'' , and the weight of 100; while the rod AC will also be in equilibrium under the two forces R'' at its ends and 20 at its centre. Hence the conditions of equilibrium of both bars and pins are satisfied in this way, and therefore we have the correct solution of the problem. Of course in this case

the bars are subject to a shearing stress and bending moment (see Chapter VIII.), which is not the case if they are weightless.

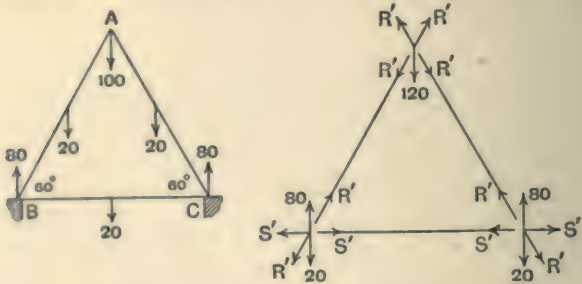


FIG. 125.

In designing a framework the stresses for the expected loads may be calculated first on the assumption of weightless bars, and the sections of the bars determined to suit the tensions and thrusts. A second approximation may then be made if necessary by calculating the stresses again, allowing for these weights, and the proposed dimensions may be adjusted accordingly, and so on, to any number of approximations.

83. Graphical Statics of Weightless Frames.

We have seen how to deal graphically with forces meeting at a point, and evidently all the work for weightless frames could be easily done graphically by drawing triangles and polygons of forces for each of the points. But these figures, instead of being drawn separately, can readily be combined into a single figure.

We will confine ourselves to the case, which is the commonest in practice, where the loads, as in the examples already solved, are applied at points on the external boundary of the frame.

If we take our second example and work it out graphically, the student will have no difficulty in working out more complicated frameworks.

In the first place the external forces will be marked and calculated as before.

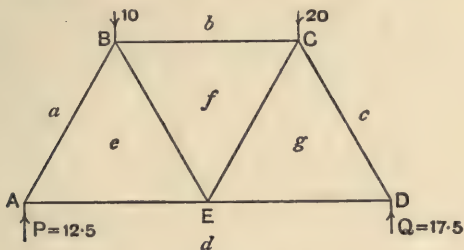


FIG. 126.

Now letter the compartments of the frame, and the compartments into which external space is divided by the external forces, with the letters a, b, c, d, e, f . In the force-diagram which we proceed to draw, the stress along a rod AB, which separates the spaces a and e , will be represented by a line $a'e'$ or $e'a'$, and so on, for the other rods.

The external forces will be marked first. As they are all parallel (in this example) they are to be drawn to lie along the same line in the force-diagram. Draw then a line to represent 30 lbs., making $a'b' = 10$ and $b'c' = 20$, and mark $c'd' = 17.5$, so that $a'b', b'c', c'd', d'a'$ represent the external forces. Next draw the triangle of forces for the point A, where we have one force known, and the directions of the three known. In other words, draw $a'e'$ parallel to AB, and $d'e'$ parallel to AE. We can now draw the polygon of forces for B, for $e'a'$ will represent one force, and $a'b'$ a second, and the polygon must be completed by lines $b'f'$ parallel to bf , and $e'f'$ parallel to ef . If we now take

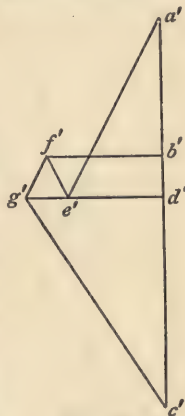


FIG. 127.

the point E we have $e'f'd'$ as vertices of the polygon for it, and it remains to draw $f'g'$ parallel to fg , and $d'g'$ parallel to dg , giving the fourth vertex g' . If we now join $c'g'$, we have the polygon of forces $b'c'g'f'$ for the point C, and the triangle $d'c'g'$ for D. The magnitudes of the stresses in each bar can now be determined at once by scaling the lines of the force-diagram. The directions of the stresses can also be determined at once from the diagram. For example, to determine the directions of the stresses of the rods meeting at C. The force-polygon for the joint C is $b'c'g'f'$ taken in this order, because the force $b'c'$ is downwards. Therefore the force $c'g'$ is from c' to g' . In other words, the force on the pin C due to the rod CD is along DC produced, and the force along CD due to the pin is towards D, or CD is in a state of thrust. Similarly, the state of any bar may be determined.

83a. As another and more complicated case take the roof truss in Fig. 127a, where in addition to the weight of the roof

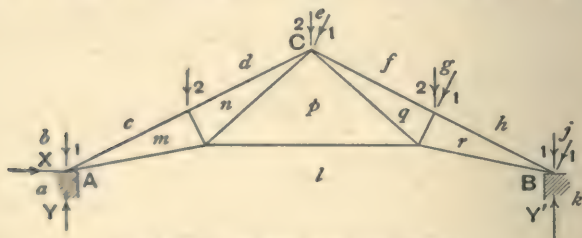


FIG. 127a.—Roof truss with wind pressure.

there is a wind pressure perpendicular to the half BC of the roof. This wind pressure is represented in the figure by forces 1, 2, 1 tons at the joints. The weight of the roof is represented in the same way by vertical forces 1, 2, 2, 2, 1, tons.

There is now a horizontal pressure at the walls, and as these horizontal reactions are in the same line there is no possibility of finding them separately. In any equation that we could

form, the two would come in together. Hence these reactions are said to be **Indeterminate**. We will make the question determinate by supposing the roof fixed at A by a pin joint, and free to move on a roller at B, so that the whole horizontal component of the wind pressure is borne at A. We will

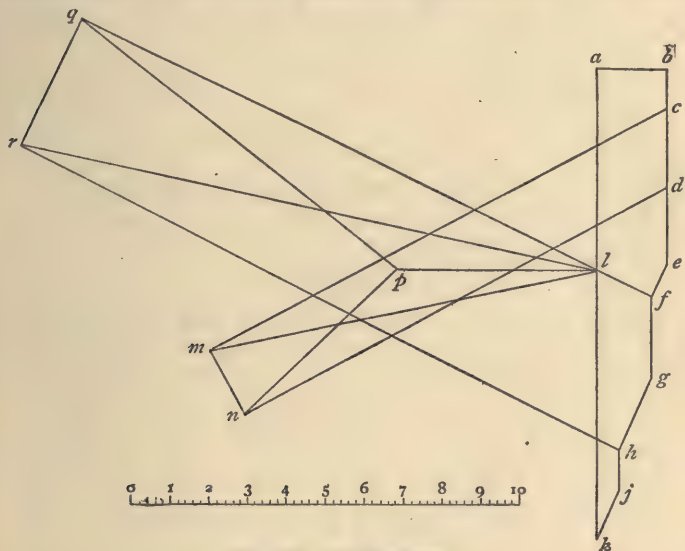


FIG. 127b.—Force diagram.

suppose the forces Y, Y' , calculated by moments for the whole frame.

$$\begin{aligned} \text{In the diagram} \quad Y &= 5.12, \\ Y' &= 6.46. \end{aligned}$$

We can now draw the force diagram for the external forces $bcdefghklab$ in Fig. 127b. X can be found from the length ab , or deduced from the horizontal component of the wind pressure.

Now start at A and draw the polygon of forces for the joint as in Art. 83. It is $abcmla$. Continue the process as before

for the different joints, and we have the stress diagram as in Fig. 127*b*. The stresses in the bars can now be read off and are given in Fig. 127*c*, where the thrusts are positive, the tensions negative.

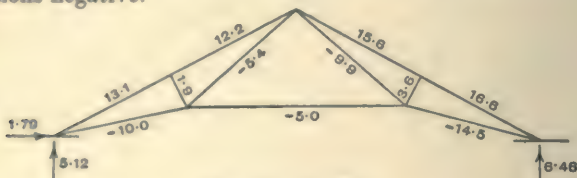


FIG. 127*c*.—Results.

Notice in the stress diagram the point *l* falls on or very close to the line *f**q*. This is only accidental, and often happens in the diagrams.

EXAMPLES.

Find the stresses in the frameworks shewn in Examples 1 to 14, both by calculation and graphically. (The bars are lettered for convenience in the answers.)

1. Warren girder, all angles being 60° .

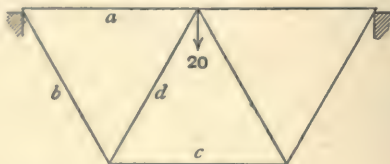


FIG. 128.

2. Girder as in No. 1, but unsymmetrically loaded.

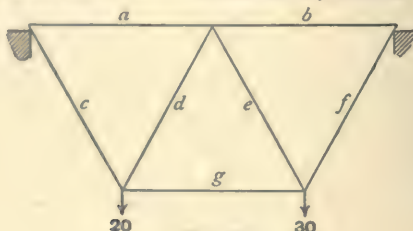


FIG. 129.

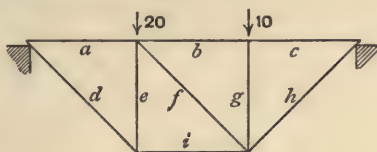
3. N girder, angles 45° and 90° .


FIG. 130.

4. Roof-truss.

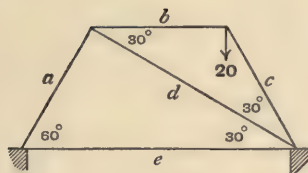


FIG. 131.

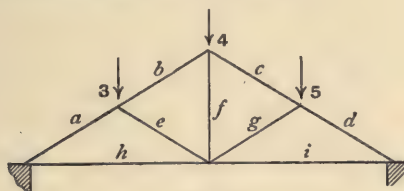
 5. Roof-truss with angles of 30° and 60° .


FIG. 132.

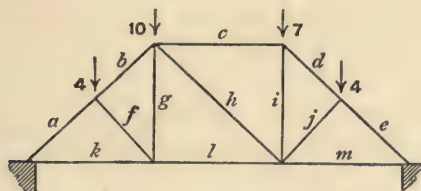
 6. Roof-truss with angles of 45° and 90° .


FIG. 133.

7. Triangular framework with bars of lengths 3, 4, 5 feet.

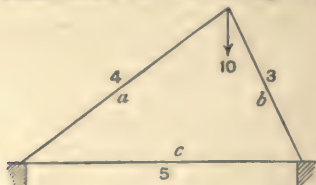


FIG. 134.

8. Triangular framework as in No. 7, but with the lengths of the bars 5, 6, 7.

9. Cantilever fixed at A and B. Angles 45° and 90° .



FIG. 135.

10. Cantilever fixed at A and B. Lengths of bars as given in the figure.

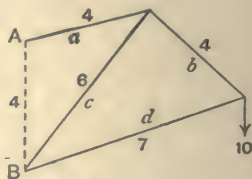


FIG. 136.

11. Cantilever of isosceles triangles each with vertical angle 30° , fixed at A and B.

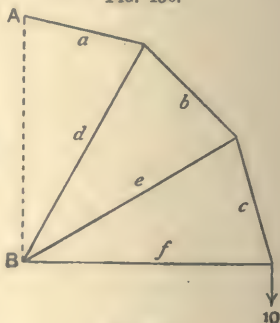


FIG. 137.

12. Cantilever fixed at A and B, angles as shewn.

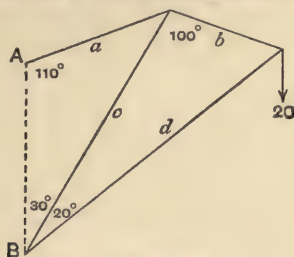


FIG. 138.

13. Crane fixed at A and B.

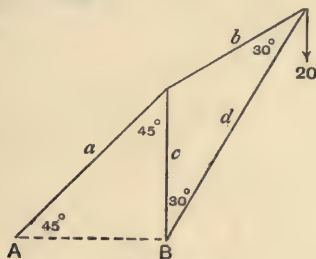


FIG. 139.

14. Crane fixed at A and B.

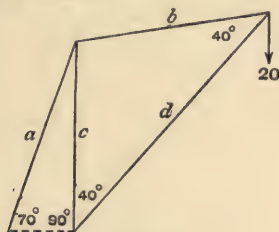


FIG. 140.

15. A bracket consisting of two rods, AB, BC, of weights 4 and 6 lbs., hinged together at B, is fixed at A and C to points in a vertical wall. If AB is horizontal and BC makes an angle 30° with AB, find the reactions at A, B, C when a weight of 10 lbs. is hung from the hinge. (Treat the pin at B separately as supporting the weight.)

16. Two rods, AB, AC, of equal length but of weights 10 and 30 lbs., hinged together at A, stand in a vertical plane with the ends on a horizontal plane sufficiently rough to prevent the rods from slipping. If a weight of 50 lbs. is hung from the hinge, find the reactions at A, B, C if the rods make an angle 60° with the vertical.

17. ABC is a light framework suspended from, and free to turn about A, and weights W_1 and W_2 are suspended from B and C. If BAC is a right angle and AB makes an angle θ with the vertical, prove that

$$W_1 \tan \theta = W_2 \tan B,$$

and find the stress in BC.

18. A smoothly jointed weightless quadrilateral is acted on by equal and opposite couples L on the opposite sides AB, CD. Find the position of equilibrium and the stresses at the joints.

CHAPTER VIII.

SHEARING STRESS AND BENDING MOMENT.

84. It has been seen that in a stretched string or in a rod, if we consider any section, there is a stress exerted between the two portions on each side of the section, so that the portion, say on the right of the section, exerts a force on the portion on the left, and the portion on the left exerts an equal and opposite force on the portion on the right. In the case of the string and the weightless frameworks this mutual stress consists simply of a tension or thrust in the direction of the length.

But in other cases, when the external forces acting on the rod are not tensions or thrusts in the direction of the length, the stresses brought into play at each section will also be more complicated. The same principle is applied as to the string in Art. 7, namely that if we take any portion of the rod, it will be in equilibrium under the stresses exerted on its ends by the adjoining portions of the rod, and any external forces such as its weight which may act on it.

Thus, if we consider a plane section AB of a rectangular bar perpendicular to its length, we see that at any element P



FIG. 141.



FIG. 142.

of the section there will be equal and opposite forces acting on the portions of the bar on each side of the section. If we

confine ourselves to the forces acting on one side (which we will call the right hand) of the section, the force at P may be resolved into two components, one perpendicular to the section, and the other in it.

The forces perpendicular to the section on the different elements, like P , can be compounded into a single force T at some point, which again may be replaced by a force T at the centre of the section and a couple M . Of course either T or M may be zero.

T is the tension or thrust, as the case may be.

M is called the **Bending Moment**.

There will, of course, be the equal and opposite force T and couple M acting on the left-hand portion of the bar.

Similarly, if we take component forces in the section, these can also be compounded into a single force S acting at the centre of the section and a couple N .

S is called the **Shearing Stress**.

N is called the **Torsion Couple**.

As before, there are an equal and opposite force and couple S and N acting on the left-hand portion.

We will not deal with any questions involving a torsion couple, as they do not occur in ordinary statical questions on frameworks, but a knowledge both of the shearing stress and bending moment, as well as of the tension or thrust, is important in deciding on the size and shape of cross section of a bar in a framework designed for any particular purpose.

85. Let us now consider one or two very simple cases to illustrate the method of obtaining the shearing stress and bending moment.

Example. 1. A rod of negligible weight is supported at its ends A , B , and carries a weight W at a point C , where $AC = a$, $BC = b$.

While the rod actually bends with the weight, the bending is usually slight, and in calculating the stresses this bending may be neglected and the rod treated as straight.

To find the shearing stress and bending moment at any point P, consider the portion AP of the rod, and let $AP = x$.

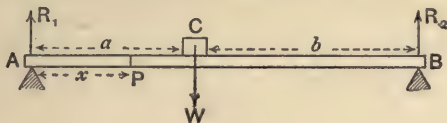


FIG. 143.

The forces acting on it are the pressure R_1 at A, and the stresses at P, which must consist therefore of a force S downwards equal to R , and a couple M equal to R_1x . There is evidently no tension in the bar.

Now

$$R_1 = \frac{a}{a+b} W;$$

$$\therefore \text{the shearing stress at } P = S = \frac{b}{a+b} W,$$

$$\text{and the bending moment at } P = M = \frac{bW}{a+b} x.$$

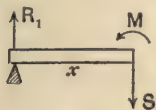


FIG. 144.

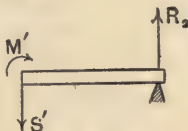


FIG. 145.

If we take a point Q between C and B, we can take either the portion AQ or QB in order to find the stresses. If we take QB and call the force and couple at Q, S' and M' , and $QB = x'$, then

$$S' = R_2 = \frac{a}{a+b} W,$$

$$M' = \frac{aW}{a+b} x',$$

and the stresses on AQ are of the same magnitude in the opposite directions.

We may draw shearing-stress and bending-moment diagrams to represent the results.

Thus in Fig. 146 the shearing stress is represented by lines in the direction in which it actually acts on the left-hand portion.

As seen from the work above, from A to C the shearing stress in this portion is downwards and equal to $aW/(a+b)$ everywhere. From C to B it is upwards and equal to $bW/(a+b)$. Hence it suddenly changes at C by an amount equal to the weight concentrated there.

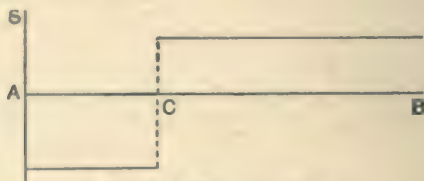


FIG. 146.—Shearing stress.

The bending moment on the left-hand portion is, however, always in the same direction and proportional to the distance from one end or the other, according as the point where the bending moment is required is on one side or the other of C.

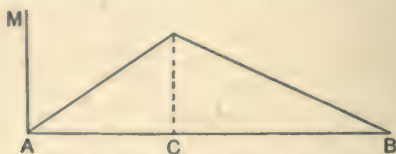


FIG. 147.—Bending moment.

If the length of the bar is $2l$ or $a+b=2l$, and x is measured from A always, we can write the expressions for the bending moment thus :

$$\text{The bending moment} = \frac{W}{2l} (2l-a)x \quad \text{if } x < a,$$

$$\text{and} \quad = \frac{W}{2l} a(2l-x) \quad \text{if } x > a.$$

Where $x=a$ these two expressions are the same. There is no sudden change in magnitude or direction of the bending moment as there is in the shearing stress, but the expression representing it suddenly changes.

In all cases of finding the shearing stress and bending moment the method is the same,—we must consider a portion of the bar from one end to the point where it is wanted, and take into account all the forces acting on the portion.

The most important quantity generally wanted is the maximum bending moment at any point of a bar.

Example. 2. A heavy uniform beam is supported at its two ends and unloaded. Obtain the shearing-stress and bending-moment diagrams.



FIG. 148.

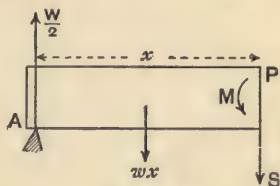


FIG. 149.

Let $2l$ = the length of the beam.
 w = the weight per unit length.
 $W = 2lw$ = the total weight.

The reactions at the supports are each $\frac{W}{2}$.

If we take a portion AP (Fig. 149) as before, we have the weight of the portion AP to take into account, and therefore the equations are:

vertically, $S + wx = \frac{W}{2};$

moments about A, $M = Sx + wx \frac{x}{2};$

$$\begin{aligned}\therefore S &= \frac{W}{2} - wx \\ &= w(l - x) \\ &= \frac{W}{2l}(l - x),\end{aligned}$$

and

$$\begin{aligned}M &= \frac{W}{2l}(l - x)x + w \frac{x^2}{2} \\ &= \frac{W}{2l}(lx - x^2) + \frac{W}{4l}x^2 \\ &= \frac{W}{2l}\left(lx - \frac{x^2}{2}\right).\end{aligned}$$

These expressions will hold from one end to the other, there being no sudden change in the value of S , or in the form of the expression for M , as there was when there was a load concentrated at one point.

The shearing stress and bending-moment diagrams drawn as before will be as follows :



FIG. 150.—Shearing stress.



FIG. 151.—Bending moment.

The shearing stress vanishes at the middle of the bar. The bending moment vanishes at each end, and is a maximum at the middle, the maximum value being $\frac{Wl}{4}$.

Example. 3. The same bar as in Question 2, but loaded with a weight W_1 at a point C, where

$$AC = a,$$

$$CB = b,$$

$$a + b = 2l.$$

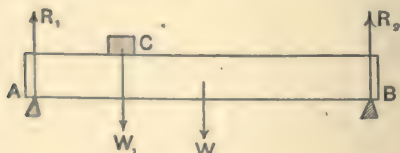


FIG. 152.

We now find

$$R_1 = \frac{W}{2} + \frac{bW_1}{2l},$$

and the student will easily find that the shearing stress and bending moment are each the sum of two parts, one due to the load only and the same as in Example 1, and the other due to the weight of the bar and the same as in Example 2.

Thus the results are :

The shearing stress at a distance x from A is

$$\frac{W}{2l}(l-x) + \frac{W_1}{2l}(2l-a) \quad \text{if } x < a,$$

$$\text{but} \quad = \frac{W}{2l}(l-x) - \frac{W_1 a}{2l} \quad \text{if } x > a;$$

$$\text{and the bending moment} = \frac{W}{2l}\left(lx - \frac{x^2}{2}\right) + \frac{W_1}{2l}(2l-a)x \quad \text{if } x < a,$$

$$\text{but} \quad = \frac{W}{2l}\left(lx - \frac{x^2}{2}\right) + \frac{W_1}{2l}a(2l-x) \quad \text{if } x > a.$$

In the same way, if we have a number of loads concentrated at points on the beam, the shearing stress and bending moment due to each separately may be calculated and added to the stress and moment due to the weight of the beam itself to give the total effect.

If we take a special case of the above when the load is equal to the weight of the beam and placed at the middle point, we have

$$W_1 = W,$$

$$ax = l.$$

$$S = \frac{W}{2l}(2l-x) \quad \text{if } x < l \quad \dots\dots\dots(i)$$

$$\text{and} \quad = -\frac{Wx}{2l} \quad \text{if } x > l. \quad \dots\dots\dots(ii)$$

$$M = \frac{W}{2l}\left(2lx - \frac{x^2}{2}\right) \quad \text{if } x < l \quad \dots\dots\dots(iii)$$

$$\text{and} \quad = \frac{W}{2l}\left(2l^2 - \frac{x^2}{2}\right) \quad \text{if } x > l. \quad \dots\dots\dots(iv)$$

Example. Draw the shearing-stress and bending-moment diagrams for this case.

86. To draw the bending-moment diagram for two concentrated loads on a weightless bar.

Let W, W' be the loads at Q and R and ADB, AEB the bending-moment diagrams for the separate loads. Then, to get the diagram for the two together, we have to draw the graph whose ordinates are the sums of the ordinates of the two. Now, as the two graphs consist of straight lines, the compound

graph will also consist of straight lines, and we can draw it at once by first marking the ordinates QH and RK , and joining AH , HK , KB .

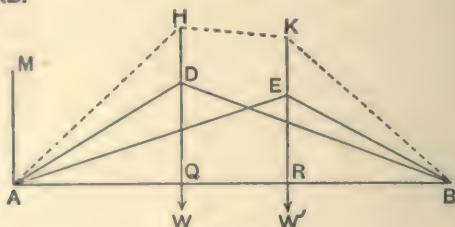


FIG. 153.

It is obvious that the maximum bending moment in this case will be at one of the two points Q or R .

The same method will apply to any number of concentrated loads.

In finding the maximum bending moment when there are concentrated loads it is especially necessary to pay attention to the ranges within which the different expressions are valid. For example, in Question 3 we have two expressions for M , (iii) holding if x lies between 0 and l , and (iv) if x lies between l and $2l$. It will be found that in (iii) M continually increases from 0 to $\frac{3}{4}Wl$, and in (iv) decreases continually from $\frac{3}{4}Wl$ to 0, so that the maximum bending moment occurs when $x=l$, though $x=l$ does not give a maximum of (iii) in the ordinary sense, the maximum of (iii) occurring when $x=2l$, which is beyond the range within which (iii) is valid.

87. As we have said, it is very important to be able to find the maximum bending moment in any case, as the greater the bending moment the more likely the beam is to break by bending.

Often we have a load that moves along the beam, and it is then necessary to find the greatest bending moment for all positions of the load.

If we take the simplest case of a single moving (or live) load W on a weightless beam of length $2l$, we have seen that when the load is at a distance a from one end the bending moment is greatest where the load is, and is there equal to $\frac{W}{2l} a(2l - a)$.

If the load moves, a varies, and the greatest value will be given by making this expression a maximum for different values of a . The maximum value is when $a = l$, or the load is at the middle of the beam, the maximum value being $\frac{Wl}{2}$.

88. The case when there are two concentrated loads moving along and keeping at the same distance from one another is a very important one, as it is the case of a travelling crane on a girder and many similar cases.

Suppose these two loads are equal, each being W , and that they remain always at a distance c apart. We will first work out the maximum bending moment when one of them is at a distance a from the end A , and the other is therefore at a distance $a + c$.

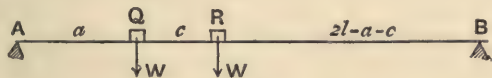


FIG. 154.

We will neglect the weight of the beam, and we have seen that in this case the maximum bending moment is either at Q or R , the points where the concentrated loads are placed. Let us calculate then the bending moment at each of these points.

Calling the moments M_Q and M_R , we have

$$\begin{aligned} M_Q &= \frac{W}{2l} a(2l - a) + \frac{W}{2l} a(2l - a - c) \\ &= \frac{W}{2l} (4al - 2a^2 - ac). \end{aligned}$$

$$\begin{aligned}
 M_R &= \frac{W}{2l} a(2l - a - c) + \frac{W}{2l} (a + c)(2l - a - c) \\
 &= \frac{W}{2l} (2a + c)(2l - a - c) \\
 &= \frac{W}{2l} (4al - 2a^2 + 2lc - 3ac - c^2). \dots\dots\dots(i) \\
 M_Q - M_R &= \frac{W}{2l} (2ac - 2lc + c^2) \\
 &= \frac{W}{2l} c(2a - 2l + c) \\
 &= \frac{W}{2l} c\{a - (2l - a - c)\}. \\
 \therefore M_Q > M_R &\text{ if } a > 2l - a - c, \\
 &\text{i.e. if } AQ > RB.
 \end{aligned}$$

In other words, the maximum bending moment is at that one of the points Q or R which is nearest to the centre of the beam.

Now, as the load travels from A towards B, R is nearer to the centre until $a = l - \frac{c}{2}$. We will find when M_R is a maximum.

Now, by (i), $M_R = \frac{W}{2l} (4al - 2a^2 + 2lc - 3ac - c^2)$,

and for different values of a this is a maximum when

$$4l - 4a - 3c = 0,$$

or

$$a = l - \frac{3}{4}c,$$

this value being within the assigned limits for a (less than $l - \frac{1}{2}c$) gives the solution of the problem, and by (i)

$$\begin{aligned}
 M_{\max} &= \frac{W}{2l} (2l - a - c)(2a + c) \\
 &= \frac{W}{2l} (2l - c - l + \frac{3}{4}c)(2l - \frac{3c}{2} + c) \\
 &= \frac{W}{2l} (l - \frac{1}{4}c)(2l - \frac{1}{2}c) \\
 &= \frac{W}{l} \left(l - \frac{c}{4}\right)^2.
 \end{aligned}$$

EXAMPLES.

1. Find the shearing stress and bending moment at any point of a cantilever (a horizontal beam fixed at one end and free at the other) of weight W and loaded with a weight W' at the free end.

2. Find the shearing stress and bending moment at points on a heavy beam of length $2l$ resting symmetrically on supports at a distance $2a$ from one another.

3. A beam is 20 ft. long and weighs 12 cwt. It rests on supports at its ends. Find the maximum bending moment due to its own weight and to a load of $\frac{1}{2}$ ton at its centre.

4. With the same beam as in Question 3, find the bending moment at any point for a load of 8 cwt. at a distance of 5 ft. from one end; find also where this bending moment is a maximum, and find its maximum value.

Compare the gradients of the bending-moment curve at points on the two sides of the load and close to it.

Find also the shearing stress at any point, the points where it is a maximum and minimum, and the maximum and minimum values.

5. A beam 20 ft. long and weighing a ton is supported at the ends and loaded with concentrated loads of $\frac{1}{4}$ ton each at four points at 4, 8, 12, 16 ft. from one end. Find expressions for the shearing stress and bending moment at all points, obtaining the numerical values at 5 and 10 ft. from the end.

Draw the shearing-stress and bending-moment diagrams.

6. Find the shearing stress and bending moment at any point in a cantilever, as in Question 1, 12 ft. long and weighing 8 cwt., with a load of 4 cwt. at the free end.

Find the maximum value of each.

7. A truck of weight 4 tons, borne equally by the four wheels, has its axles 10 ft. apart, and it passes along a pair of girders 50 ft. long supported at the ends of each. What is the maximum bending moment due to the truck on one girder for any position of the truck?

Find also the bending moment at the centre and at a wheel when the truck is at the centre of the girder.

8. Draw a graph to shew the maximum bending moment in a weightless beam due to a moving load passing along it. (y to represent the max. B.M. in the beam when the position of the moving load is at x .)

9. By taking a portion of length δx of a bar, shew that where there is no concentrated load $\frac{dM}{dx} = S$.

10. Two equal uprights ABC, DEF are jointed to fixed points A, D, on the same level and braced by equal light bars BF, CE, jointed at their ends to the uprights. A force X in the direction CF is applied at C. Find the stresses in BF, CE, and shew that the maximum bending moments are $\frac{(2a-b)b}{a+b}X$ and $\frac{3ab}{a+b}X$, where AC = a, AB = b.

CHAPTER IX.

MISCELLANEOUS QUESTIONS

89. Friction.

A few additional questions involving friction may be mentioned.

Friction Circle.

When a cylindrical pin fits into a cylindrical hole of slightly greater radius, as an axle in contact with its bearings, we may suppose the pin to be in contact with the body at one point.

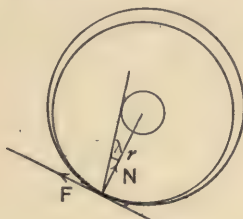


FIG. 155.

If the body is in limiting equilibrium, the resultant reaction, which acts at an angle λ with the normal, will pass at a distance $r \sin \lambda$ from the centre if r is the radius and λ the angle of friction. Thus the resultant reaction, wherever the point of contact lies, will touch a circle of radius $r \sin \lambda$ if friction is limiting, and if friction is not limiting it will cut this circle.

If a weightless bar is acted on only by the reactions at two rough pins, these reactions must be in the same line and must



FIG. 156.

meet the friction circles at the two ends. If the friction is limiting at each end the reactions will act along a common tangent to the friction circles.

90. Carriage Wheel.

Suppose a carriage drawn along by a horizontal force P applied to the axle.

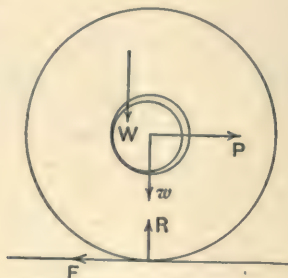


FIG. 157.—Carriage wheel.

Let W = weight of axle and weight borne by it.

w = weight of wheel, acting at its centre.

a = radius of axle.

b = radius of wheel.

When the carriage is pulled along, the radius to the point of contact of the axle with the wheel will make some angle θ with the vertical in the direction shewn in the figure, for the motion of the wheel tends to carry the axle upwards from the lowest point.

Also, let S , μS be the normal reaction and friction at the point of contact.

R , F be the normal reaction and friction at the ground.

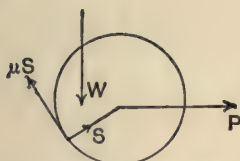


FIG. 158.—The axle.

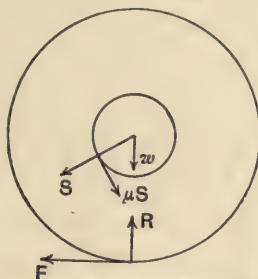


FIG. 159.—The wheel.

The wheel and axle with the forces acting on them are shewn separately in Figures 158 and 159. Forming the equations for these separately, we have

For the axle,

$$\text{horizontally, } P + S \sin \theta - \mu S \cos \theta = 0, \dots\dots\dots(1)$$

$$\text{vertically, } W - S \cos \theta - \mu S \sin \theta = 0; \dots\dots\dots(2)$$

$$\begin{aligned} \therefore \frac{P}{W} &= \frac{\mu \cos \theta - \sin \theta}{\mu \sin \theta + \cos \theta} \\ &= \frac{\frac{\sin \lambda}{\cos \lambda} \cos \theta - \sin \theta}{\frac{\sin \lambda}{\cos \lambda} \sin \theta + \cos \theta} \\ &= \frac{\sin (\lambda - \theta)}{\cos (\lambda - \theta)} \\ &= \tan (\lambda - \theta). \dots\dots\dots(3) \end{aligned}$$

For the whole system, horizontally,

$$P = F. \dots\dots\dots(4)$$

For the wheel only, moments about the centre,

$$\mu S a = F b = P b; \dots\dots\dots(5)$$

$$\begin{aligned}\therefore \text{ from (1), } \quad \mu Sa &= S(\mu \cos \theta - \sin \theta)b; \\ \therefore a \sin \lambda &= b(\sin \lambda \cos \theta - \cos \lambda \sin \theta); \\ \therefore a \sin \lambda &= b \sin (\lambda - \theta). \dots\dots\dots(6)\end{aligned}$$

(6) gives θ .

$$\text{From (3), } \quad P/W = \tan (\lambda - \theta) = \frac{\sin (\lambda - \theta)}{\cos (\lambda - \theta)};$$

$$\therefore \text{ by (6), } \quad P/W = \frac{\frac{a}{b} \sin \lambda}{\sqrt{1 - \frac{a^2}{b^2} \sin^2 \lambda}} = \frac{a \sin \lambda}{\sqrt{(b^2 - a^2 \sin^2 \lambda)}}. \dots\dots(7)$$

If $\frac{a}{b}$ is small, as is usual, from (6) $\sin (\theta - \lambda)$ is small

or $\theta \approx \lambda$,

where the symbol $\approx$ stands for "is nearly equal to," and

$$P/W \approx \frac{a}{b} \sin \lambda.$$

91. Rolling Friction.

When a body such as the carriage wheel in the last article is capable of rolling along a horizontal plane, there is an action

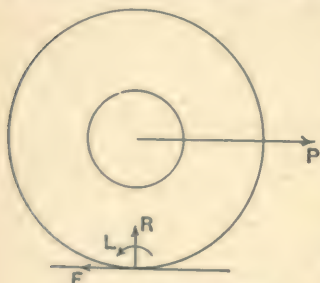


FIG. 160.—Rolling friction.

brought into play which tends to oppose the rolling. For if not, the slightest force applied horizontally would cause it to roll. In practice this is found not to be the case, shewing that there is a couple acting, due doubtless to the contact not taking place at a point (or line along the surface of the rolling cylinder), but over a small

area. This couple is called the **Friction Couple**, and may be measured by the moment about the point of contact, of the smallest force applied in a given direction which causes rolling.

The rolling friction has been neglected in the example of the carriage wheel (Art. 90). It would be represented by a small couple L acting as shewn in Fig. 160.

92. Tension of a Rope passing round a Rough Post.

This question can only be worked out conveniently by the Calculus, but as the solution is very simple it will be given here.

If a rope wraps round any angle of a cylindrical post and is on the point of slipping when pulled with a force T_1 at one end and T_0 at the other, where $T_1 > T_0$, then we may consider each little element of the rope as in limiting equilibrium.

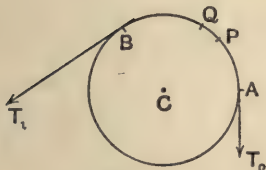


FIG. 161.

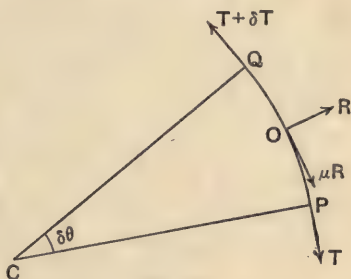


FIG. 162.

It is necessary to think of a small element PQ , which is drawn on a larger scale in Fig. 162.

Let O be the middle point of PQ ,

C be the centre of the circle PQ ,

$\delta\theta$ be the angle PCQ in circular measure.

The forces acting on PQ may be represented by

- (1) The tension T at one end P .
- (2) A slightly different tension $T + \delta T$ at Q .
- (3) A normal reaction R along CO .
- (4) Friction μR (since the equilibrium is supposed limiting) perpendicular to R and in the direction to help T if δT is positive.

The angles between the tensions and the friction will each be $\frac{\delta\theta}{2}$; hence resolving along CO,

$$R = T \sin \frac{\delta\theta}{2} + (T + \delta T) \sin \frac{\delta\theta}{2}, \dots\dots\dots(1)$$

and perpendicular to CO,

$$\mu R + T \cos \frac{\delta\theta}{2} = (T + \delta T) \cos \frac{\delta\theta}{2}. \dots\dots\dots(2)$$

Now $\sin \frac{\delta\theta}{2} \approx \frac{\delta\theta}{2}$

and $\cos \frac{\delta\theta}{2} \approx 1$

if we neglect squares and products of small quantities.

Hence we get from (1),

$$R = T \frac{\delta\theta}{2} + (T + \delta T) \frac{\delta\theta}{2} = T \delta\theta,$$

and from (2), $\mu R + T = T + \delta T;$

$$\therefore R = T \delta\theta$$

and $\mu R = \delta T;$

$$\therefore \frac{\delta T}{T} = \mu \delta\theta;$$

$$\therefore \int \frac{dT}{T} = \mu \theta + c;$$

$$\therefore \log_e T - \log_e T_0 = \mu \theta;$$

$$\therefore \log_e \frac{T}{T_0} = \mu \theta;$$

$$\therefore T = T_0 e^{\mu\theta},$$

where T_0 is the tension at A and θ is the angle PCA.

In particular, if the rope wraps on an angle α (ACB) of the post,

$$T_1 = T_0 e^{\mu\alpha}.$$

E.g. if the rope wraps once round the post and $\mu = 0.2$,

$$\mu\alpha = \frac{1}{5} \times 2\pi,$$

and

$$T_1 = 2.79T_0.$$

If the rope makes n complete turns round the post a tension $T_0 e^{\mu 2\pi n}$ is required to move the rope against a tension T_0 at the other end, and obviously this will increase very rapidly with n .

93. Screw with Friction.

Suppose we have a screw with a square thread, and that the coefficient of friction between the thread and the block is everywhere μ , other friction being negligible.

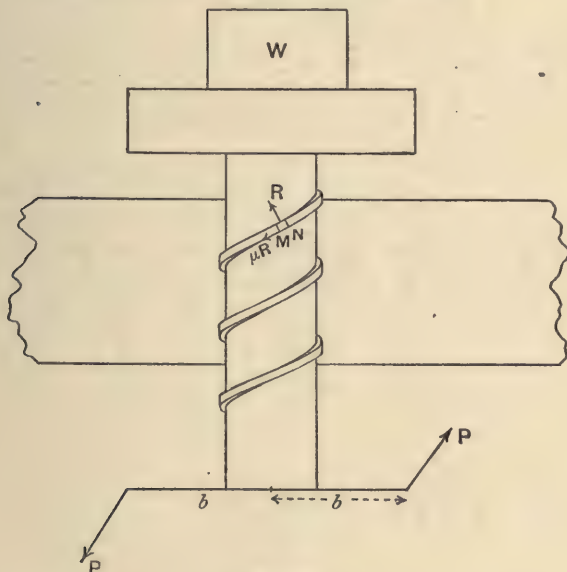


FIG. 163.

Suppose the screw vertical, as in the figure, and being used to raise a weight W .

Let b = the length of the arm to which each force is applied,
so that the screw is moved by a couple of moment $P \cdot 2b$.

p = the pitch of the screw,

α = the angle the thread makes with the horizontal (or
the angular pitch),

a = the radius of the screw,

so that $p = 2\pi a \tan \alpha$,

as will be seen by imagining the surface of the screw cut along
its length, and unrolled so as to lie in a plane, when we get
the Fig. 164.

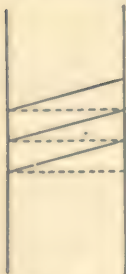


FIG. 164.

Besides the forces P and the weight W we will
have reactions acting at each little element of
the thread, between the thread and the block.
These we can denote on a particular element MN
by the normal reaction R and friction μR if the
screw is on the point of moving.

In Fig. 163 the forces P are on the point of
overcoming the friction and raising the weight.

It will be seen also that if the screw is rotated
through a small angle $\delta\theta$, any element MN will
move a distance $a \delta\theta \cdot \sec \alpha$ in the direction of the thread, and
consequently work is done against friction by the motion of MN ,

$$= \mu R a \sec \alpha \cdot \delta\theta,$$

and the total work done against friction will be

$$\Sigma \mu R a \sec \alpha \cdot \delta\theta = \mu a \sec \alpha \cdot \delta\theta \Sigma R.$$

The work done on the weight

$$= W a \delta\theta \tan \alpha,$$

for W will rise a distance $a \delta\theta \tan \alpha$.

The work done by the forces P will be

$$2Pb \delta\theta;$$

$$\therefore 2Pb \delta\theta = W a \delta\theta \tan \alpha + \mu a \sec \alpha \cdot \delta\theta \Sigma R;$$

$$\therefore 2Pb = W a \tan \alpha + \mu a \sec \alpha \Sigma R; \dots\dots\dots(1)$$

also, if we resolve the forces parallel to the axis of the screw, we get

$$\begin{aligned} W + \Sigma \mu R \sin \alpha &= \Sigma R \cos \alpha ; \\ \therefore W &= \Sigma R (-\mu \sin \alpha + \cos \alpha) \\ &= (-\mu \sin \alpha + \cos \alpha) \Sigma R ; \dots\dots\dots(2) \end{aligned}$$

$\therefore$ by (1) and (2),

$$\begin{aligned} 2Pb &= Wa \tan \alpha - \frac{\mu a \sec \alpha W}{\mu \sin \alpha - \cos \alpha} \\ &= Wa \left(\tan \alpha - \frac{\mu \sec \alpha}{\mu \sin \alpha - \cos \alpha} \right) \\ &= Wa \frac{\sin \alpha (\mu \sin \alpha - \cos \alpha) - \mu}{\cos \alpha (\mu \sin \alpha - \cos \alpha)} \\ &= Wa \frac{\mu \cos^2 \alpha + \sin \alpha \cos \alpha}{\cos \alpha (\cos \alpha - \mu \sin \alpha)} \\ &= Wa \frac{\mu \cos \alpha + \sin \alpha}{\cos \alpha - \mu \sin \alpha} \end{aligned}$$

If λ is the angle of friction, this can be written

$$\begin{aligned} 2Pb &= Wa = \frac{\frac{\sin \lambda}{\cos \lambda} \cdot \cos \alpha + \sin \alpha}{\cos \alpha - \frac{\sin \lambda}{\cos \lambda} \cdot \sin \alpha} \\ &= Wa \frac{\sin (\lambda + \alpha)}{\cos (\lambda + \alpha)} \\ &= Wa \tan (\lambda + \alpha). \\ \frac{W}{P} &= \frac{2b}{a} \cot (\alpha + \lambda). \end{aligned}$$

If λ is less than α and the screw is on the point of running back, being just held by forces P' at the ends of the arm,

$$\frac{W}{P'} = \frac{2b}{a} \cot (\alpha - \lambda).$$

If λ is greater than α , no weight, however large, can cause the screw to run back.

94. Astatics.

If a system of forces acting at given points in a body are in equilibrium, and the body is rotated through any angle about a fixed point, while the forces continue to act at the same points and in the same directions, the forces will not in general remain in equilibrium. For example, a magnet may be thought of as acted on by two equal and opposite forces at its two poles, which remain unchanged in magnitude and direction in whatever position the magnet is placed. If the magnet is capable of rotation about a vertical axis only, it will be in equilibrium when its direction coincides with the direction of the magnetic forces, and if rotated from this position and released it will not be in equilibrium, but will begin to move towards the equilibrium position.

If the forces acting as above on a body are still in equilibrium when the body is turned through any angle about a fixed point, they are said to be in **Astatic Equilibrium**. Such a case occurs when two magnets of equal length and strength are connected so as to be parallel to one another, but with their poles in opposite directions. The magnets form an astatic combination.

Suppose that any forces whose components are $X_1, Y_1; X_2, Y_2; \text{etc.}$, act at points whose coordinates referred to a point O in the body are $x_1, y_1; x_2, y_2; \text{etc.}$, and we wish to discover whether any rotation of the body about O , while the forces act at the same points in the body and in the same directions in space, will cause the body to be in a state of equilibrium, and if this is the case to find whether more than one such position of equilibrium exists, and what conditions are necessary that all such positions are positions of equilibrium.

First examine the effect of a rotation θ about O on a single force. The components of the force remain unchanged, but the point of application changes from P to P' in Fig. 165.

This makes the new moment about O,

$$\begin{aligned}
 & Y_1 r_1 \cos (\theta + \alpha_1) - X_1 r_1 \sin (\theta + \alpha_1) \\
 &= Y_1 r_1 (\cos \theta \cos \alpha_1 - \sin \theta \sin \alpha_1) \\
 &\quad - X_1 r_1 (\sin \theta \cos \alpha_1 + \cos \theta \sin \alpha_1) \\
 &= Y_1 (x_1 \cos \theta - y_1 \sin \theta) - X_1 (x_1 \sin \theta + y_1 \cos \theta) \\
 &= (Y_1 x_1 - X_1 y_1) \cos \theta - (Y_1 y_1 + X_1 x_1) \sin \theta.
 \end{aligned}$$

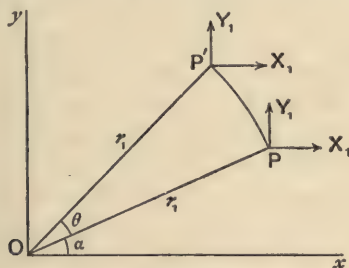


FIG. 165.

Hence, after such a rotation the forces are equivalent to a force at O, whose components are ΣX and ΣY , and a couple whose moment is

$$\cos \theta \Sigma (Yx - Xy) - \sin \theta \Sigma (Xx + Yy) = G \cos \theta - V \sin \theta,$$

where $G = \Sigma (Yx - Xy)$

= sum of the moments of the forces about O in their original positions,

and $V = \Sigma (Xx + Yy).$

The forces can only be in equilibrium if

$$(i) \Sigma X = 0 \text{ and } \Sigma Y = 0,$$

$$\text{and} \quad (ii) \quad G \cos \theta - V \sin \theta = 0.$$

(i) shews that the forces in their original positions must have been equivalent to a couple or in equilibrium, and (ii) gives

$$\tan \theta = G/V,$$

giving two values of θ differing by 180° , which make the couple in the altered position zero.

The equilibrium will be astatic if, for all values of θ ,

$$G \cos \theta - V \sin \theta = 0,$$

which requires that $G=0$ and $V=0$,

or
$$\Sigma(Yx - Xy) = 0$$

and
$$\Sigma(Xx + Yy) = 0.$$

95. Astatic Centre of Forces.

If, instead of rotating the body and letting the forces remain in the same direction, we keep the body fixed and rotate the forces through an angle $-\theta$, we will evidently get the same result for the moment of the forces, but with the advantage that the axes are now fixed in the body. The component forces are, however, altered to

$$\Sigma(X \cos \theta + Y \sin \theta) = \cos \theta \Sigma X + \sin \theta \Sigma Y$$

and
$$\Sigma(Y \cos \theta - X \sin \theta) = \cos \theta \Sigma Y - \sin \theta \Sigma X,$$

as may be seen at once from Fig. 166, which shews the old and new positions of the components of one of the forces.

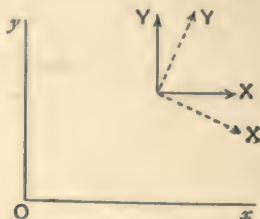


FIG. 166.

Now we have seen (Art. 70) that the line of action of a number of forces is given by

$$\eta \Sigma X - \xi \Sigma Y + G = 0,$$

and hence, when the forces have been turned round in this way, the line of action will be given by

$$\eta(\cos \theta \Sigma X + \sin \theta \Sigma Y) - \xi(\cos \theta \Sigma Y - \sin \theta \Sigma X) + G \cos \theta - V \sin \theta = 0$$

or
$$(\eta \Sigma X - \xi \Sigma Y + G) \cos \theta + (\eta \Sigma Y + \xi \Sigma X - V) \sin \theta = 0.$$

This equation shews that whatever the value of θ , the resultant force will pass through the definite point given by

$$\eta \Sigma X - \xi \Sigma Y + G = 0,$$

$$\eta \Sigma Y + \xi \Sigma X - V = 0.$$

This point is called the **Astatic Centre** of the forces. The astatic centre for any forces may be compared with the centre of a system of parallel forces ; in fact, the case of parallel forces is only a special case of the above.

Example. Prove that if the forces reduce to a couple the astatic centre is at infinity.

96. Questions on the **Minimum Amount of Material required in a Frame.**

Example. 1. A weight W is to be supported at some point A in a given vertical line by means of two weightless rods attached at given points B and C in the same horizontal, A, B, C being in a vertical plane. If the sections of the rods are proportional to their tensions, find the depth of A below BC when the total amount of material used in the rods is least.

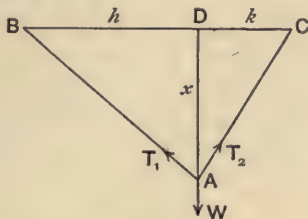


FIG. 167.

Let AD be the vertical from A to BC , and let

T_1, T_2 be the tensions.

h, k be the given lengths BD, DC .

$x = AD$, the required depth of A .

Then, by Lami's Theorem,

$$\frac{T_1}{\cos C} = \frac{T_2}{\cos B} = \frac{W}{\sin A},$$

and if the cross sections are mT_1 and mT_2 , the volume of material used is

$$\begin{aligned}
 & mT_1c + mT_2b \\
 &= \frac{mW}{\sin A} (c \cos C + b \cos B) \\
 &= \frac{mW}{\sin A} \left\{ \frac{ck}{b} + \frac{bh}{c} \right\} = mW \frac{c^2k + b^2h}{bc \sin A} \\
 &= mW \frac{k(x^2 + h^2) + h(x^2 + k^2)}{x(h+k)} \\
 &= mW \frac{x^2 + hk}{x} \\
 &= mW \left(x + \frac{hk}{x} \right),
 \end{aligned}$$

and this is a minimum when $x^2 = hk$, or the angle BAC is a right angle.

2. A weight W is to be supported at A by two rods fixed to two points B, C , in the same vertical. If AB is horizontal and of given length h , find the length of CA so that the least quantity of material may be required for the two rods, assuming the sections proportional to the stresses in the rods and the weights of the rods themselves neglected.

If the section for a tension is made r times the section for a thrust of the same magnitude, find the length of CA for the minimum quantity of material.

3. A horizontal bar AB , loaded in any way, is hinged at A and supported by a weightless strut CD jointed to a point C vertically below A and to a point in the rod. Shew that if the section of CD is proportional to the thrust in it, the quantity of material in it will depend only on the inclination of CD to the vertical, not on its length, and will be least when the strut makes an angle 45° with the vertical.

EXAMPLES.

1. A weight W is raised by a string passing round a pulley turning on a rough axle. If P is the force exerted and the radius of the axle is b , and that of the wheel is a , and the angle of friction is λ , prove that

$$\frac{W}{P} = \frac{a - b \sin \lambda}{a + b \sin \lambda}.$$

2. In a rough screw verify that the work done by the applied force is the sum of the works done against the opposing force and against friction.

CHAPTER X.

CENTRES OF MASS (or CENTRES OF GRAVITY).

97. If we think of two particles of weights w_1 and w_2 at A and B, and attached to one another by a weightless rigid wire, the forces due to gravity are equivalent to a single force

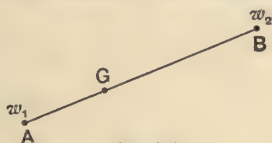


FIG. 168.

acting at a point G, dividing AB in the ratio of $w_2 : w_1$. It will not matter in what position the wire is held, the resultant will always pass through the same point.

Similarly, if we take three particles w_1, w_2, w_3 at A, B, C, attached to a light rigid framework so as always to retain

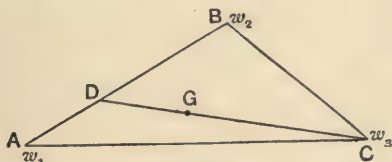


FIG. 169.

the same distances from one another, the parallel forces w_1, w_2, w_3 can be compounded into a single force by first compounding w_1 and w_2 into a single force $w_1 + w_2$ at D, where

$AD : DB = w_2 : w_1$, and then compounding $w_1 + w_2$ and w_3 into a single force at G, where $CG : GD = w_1 + w_2 : w_3$.

As the forces have a definite resultant, the same point G must be obtained in whatever order we proceed with the composition.

As in the case of two particles, it will be seen that it does not matter in what position the framework is held, the resultant will always pass through the same point in the triangle. We will have the same results for any number of particles, and we will consequently define the Centre of Mass or Centre of Gravity as follows :

The **Centre of Mass** of a number of particles is a point through which the resultant of parallel forces on the particles proportional to their masses, acts, in whatever position the particles are placed, provided that they keep at the same relative distances from one another.

Since any rigid body can be thought of as made up of particles, it has a centre of mass defined in this way.

It is to be noticed that for ordinary bodies the forces due to gravity act as nearly as possible in the manner expressed in the definition, namely parallel to one another and proportional to the masses, and hence we can speak justly of the centre of gravity. But if the body is very large, such as a mountain, the forces due to gravity are not accurately proportional to the masses, nor are they accurately parallel, and in fact the forces due to gravity may not compound into a single force at all, and consequently the term Centre of Gravity would be a misnomer. On the other hand, a definite centre of mass is given by the definition in such a case as in any other. Hence modern writers generally adopt the term Centre of Mass in preference to Centre of Gravity.

98. Centre of Mass by Experiment.

If we support a rigid body at one point and let it hang at rest, it will be acted on only by gravity, and by the reaction

at the point of support C . The resultant effect of gravity is a single vertical force through the centre of mass G . This must be balanced by the reaction at C , and therefore C and G must be in the same vertical line.

We thus have a ready means of finding the centre of mass (which we will often abbreviate into C.M.) in some cases. Thus, if the body is an irregular thin sheet of card, we can support it by a pin at one point, so that it hangs in a vertical plane, and mark the vertical through the pin. Then support it at a

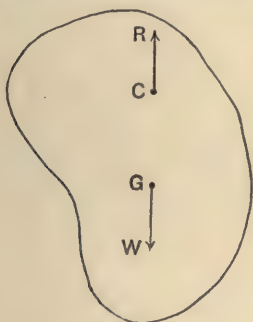


FIG. 170.

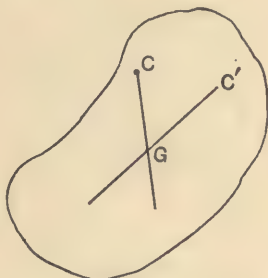


FIG. 171.

second point and do likewise. The centre of mass will be in the card on the perpendicular to the surface at the point where the lines marked intersect.

We can get the position of the C.M. roughly, in the same way, for many irregular bodies, such as a chair or waste-paper basket, and it will be noticed that the C.M. need not be in the material of the body. In the waste-paper basket the C.M. will be found in the interior of the basket.

99. Centre of Mass by Calculation.

In most cases where the C.M. is required, it is found by first dividing the body up into pieces whose C.M.'s are known, and then deducing the C.M. of the whole.

The C.M. of a thin uniform rod is evidently at its middle point.

100. C.M. of a thin uniform plate or lamina (e.g. a thin sheet of metal) in the shape of a parallelogram.

This may, for shortness, be called the **C.M. of a parallelogram.**

Let ABCD be the parallelogram. It may be divided into an infinitely large number of infinitely thin strips parallel to a side AD. The C.M. of each strip is at its middle point; in

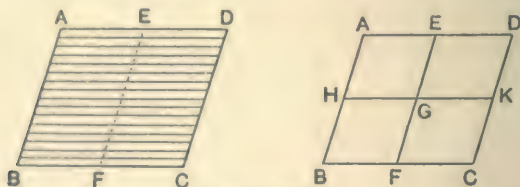


FIG. 172.

other words, the resultant of the forces on a strip passes through the middle point of the strip. All these resultants form a set of parallel forces acting at points on the line EF in Fig. 172. The resultant of all these must, therefore, act through some point on EF; or, in other words, the C.M. of the whole parallelogram is somewhere on EF. Similarly, the C.M. lies on the line HK bisecting AB and DC. Hence it lies on the intersection of these two lines, which is also the intersection of the diagonals of the parallelogram.

101. C.M. of a Parallelopiped.

As in the last article, the parallelopiped may be divided into parallel plates, and the C.M. will be found to be at the intersection of the lines joining the middle points of opposite faces, which is also the intersection of the diagonals.

102. C.M. of a Thin Uniform Triangular Plate (or C.M. of a Triangle).

The triangle ABC may be divided into an infinite number of infinitely thin strips, the centres of mass of which all lie on the median AD of the triangle, and therefore the C.M. of the whole triangle lies on AD. Similarly the C.M. lies on the median BE, and is therefore at the intersection of these medians.

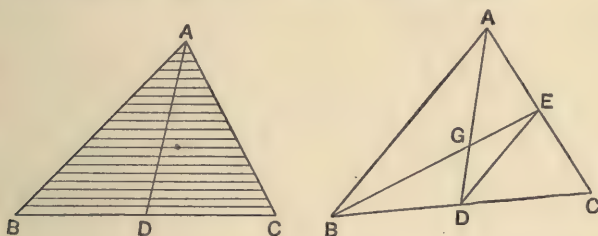


FIG. 173.

As there is only one centre of mass, the well-known geometrical result follows at once, that the three medians meet in one point.

Note also that by similar triangles

$$\frac{AG}{GD} = \frac{AB}{DE} = 2$$

or $GD = \frac{1}{3}AD.$

Note further that if we try to find the C.M. of three equal particles at the vertices of the triangle, we proceed by combining masses m and m at B and C into $2m$ at D, and then combine $2m$ at D with m at A, giving the centre of mass at a point dividing AD in the ratio 2 : 1. This coincides with the C.M. of the triangular plate. Hence a uniform triangular plate has the same centre of mass as three equal particles placed at its vertices.

103. C.M. of a Tetrahedron.

Let the tetrahedron ABCD be divided into a series of infinitely thin plates such as PQR by planes parallel to the face BCD.

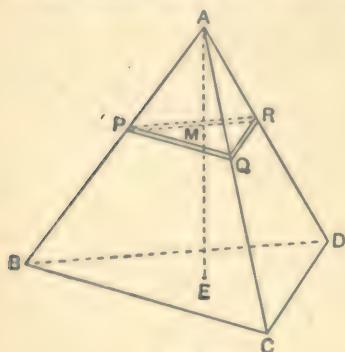


FIG. 174.

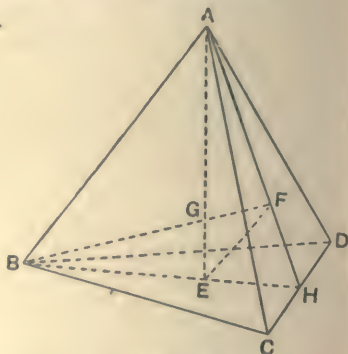


FIG. 175.

Let the centre of mass of PQR be at M. It will easily be seen by similar triangles that M is on the line joining A to E, the C.M. of the triangle BCD. Hence the centres of mass of all the plates are on the same line, and therefore this line contains the centre of mass of the tetrahedron. Similarly the C.M. is on the line from B to F, the C.M. of the triangle ACD. These lines therefore intersect, and the point of intersection G is the C.M. of the tetrahedron.

That AE and BF intersect may be seen at once geometrically by bisecting CD in H. Then E and F are on BH and AH, and therefore BF and AE are in the plane AHB, and therefore intersect.

Again, on joining EF we have

$$HE = \frac{1}{3}HB$$

and

$$HF = \frac{1}{3}HA;$$

$$\therefore EF = \frac{1}{3}AB;$$

also

$$\frac{AG}{GE} = \frac{AB}{EF} = 3,$$

$$\therefore EG = \frac{1}{4}EA.$$

The C.M. of the tetrahedron coincides with the C.M. of four equal particles placed at its vertices.

For three equal particles, each of mass m at B, C, D, have their C.M. at E (Art. 102), and the C.M. of the four is the same as that of a mass $3m$ at E and m at A, which therefore is on AE, and divides AE in the ratio of 3 : 1.

104. Centre of Mass of a Compound Body.

If we have the masses and centres of mass of each of two portions of a body, we can find the centre of mass of the whole.

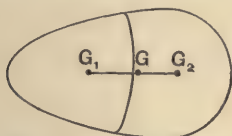


FIG. 176.—Compound body.

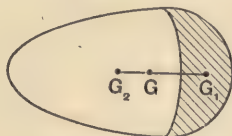


FIG. 177.—Remainder.

Let m_1, m_2 be the masses of the two portions,

G_1, G_2 their centres of mass.

The C.M. of the whole will be at G , where

$$\frac{G_1G}{GG_2} = \frac{m_2}{m_1}$$

or

$$\frac{G_1G}{m_2} = \frac{GG_2}{m_1} = \frac{G_1G_2}{m_1 + m_2};$$

$$\therefore G_1G = \frac{m_2}{m_1 + m_2} G_1G_2.$$

105. Centre of Mass of a Remainder.

Similarly, if we know the mass of a body and the position of its C.M., and we cut off from it a portion of known mass and C.M., we can readily find the C.M. of the remainder.

For if (Fig. 177) G is the C.M. of the whole, whose mass is M , and G_1 the C.M. of the part (of mass m) cut off, then G_2 , the C.M. of the remainder, will be in G_1G produced, so that

$$\frac{G_2G}{GG_1} = \frac{m}{M - m}.$$

106. C.M. of any Quadrilateral.

If the C.M. of any rectilineal area is required, it is generally necessary to divide it up into triangles, and use the method of Art. 104. In the case of a quadrilateral of any shape, a simple geometrical construction for the C.M. can be obtained in the following way.

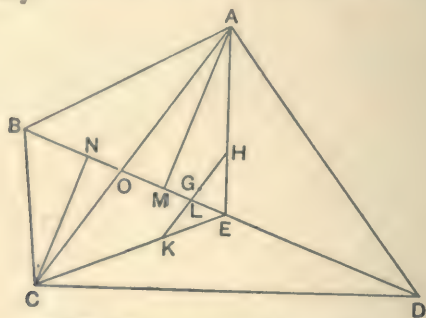


FIG. 178.—Centre of mass of a quadrilateral.

Let ABCD be the quadrilateral,

O the intersection of the diagonals,

E the middle point of BD,

H, K the C.M.'s of the triangles ABD, CBD,

so that

$$HE = \frac{1}{3}AE,$$

$$KE = \frac{1}{3}CE.$$

It remains to divide HK inversely in the ratio of the areas of these triangles.

Now, if AM, CN are the perpendiculars from A and C on BD, the areas are proportional to AM and CN, and therefore to AO and CO. Hence also, if HK cuts BD in L, .

$$\frac{HL}{LK} = \frac{AO}{OC} = \frac{ABD}{CBD},$$

or L divides HK in the ratio of the areas of the triangles.

Hence, if we mark off $HG = KL$, G will divide HK in the inverse ratio of the areas, and therefore G is the C.M. of the quadrilateral.

107. Centre of Mass of a Number of Particles.

Let there be a number of particles $m_1, m_2, m_3, \dots$ in one plane, their positions being given by the coordinates $x_1, y_1; x_2, y_2; \dots$ etc.

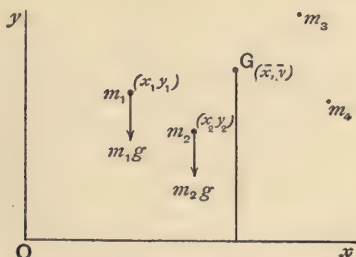


FIG. 179.

Place the plane vertical with Oy vertical, and think of the particles as acted on by their weights $m_1g, m_2g, \dots$ proportional to their masses and parallel to Oy , and at distances $x_1, x_2, \dots$ from O . Hence (by Art. 33) the resultant of these forces acts along a line at a distance $\bar{x}$ from O given by

$$\bar{x} = \frac{\sum mgx}{\sum mg} = \frac{\sum mx}{\sum m};$$

consequently the C.M. is somewhere on this line, or its x -coordinate is

$$\bar{x} = \frac{\sum mx}{\sum m}.$$

By turning the plane round so that Ox is vertical, we get $\bar{y}$, the y -coordinate of the C.M., and

$$\bar{y} = \frac{\sum my}{\sum m}.$$

We will see from the results of Chapter XIV. that the same formula holds if the particles are not in one plane.

If the particles are distributed anywhere in space, there are three coordinates required to fix the positions of each particle

x_1, y_1, z_1 , to denote the distances of the particle m_1 from three planes, and the distances of the C.M. from these planes will be

$$\bar{x} = \Sigma mx / \Sigma m$$

$$\bar{y} = \Sigma my / \Sigma m$$

$$\bar{z} = \Sigma mz / \Sigma m.$$

Examples. 1. Find the coordinates of the C.M. of a triangle in terms of the coordinates of the vertices.

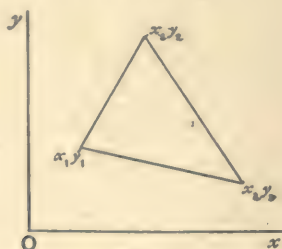


FIG. 180.

As the centre of mass of the triangle is the same as that of three equal masses at its vertices, it follows at once that if the coordinates of the vertices are x_1, y_1 ; x_2, y_2 ; x_3, y_3 ; the coordinates of the centre of mass are

$$\frac{x_1 + x_2 + x_3}{3} \quad \text{and} \quad \frac{y_1 + y_2 + y_3}{3}.$$

2. One end of a rectangular plate is cut off obliquely so as to leave a plate as in the figure. Find the distance of its C.M. from the sides a and b .

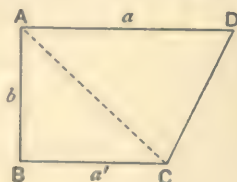


FIG. 181.

We will illustrate the work above by solving this question in four ways.

First solution. Divide it into two triangles by joining AC.

The areas of these triangles are $\frac{1}{2}ab$ and $\frac{1}{2}a'b$.

The distances from AD and AB of the C.M. of ACD are

$$\frac{b}{3} \quad \text{and} \quad \frac{a+a'}{3}$$

by Example 1 above, and those of the C.M. of ABC are

$$\frac{2b}{3} \quad \text{and} \quad \frac{a'}{3};$$

$\therefore$ the distances of the C.M. of the whole are

$$\frac{\frac{1}{2}ab \frac{b}{3} + \frac{1}{2}a'b \frac{2b}{3}}{\frac{1}{2}ab + \frac{1}{2}a'b} = \frac{\frac{ab}{3} + \frac{2a'b}{3}}{a+a'} = \frac{1}{3} \frac{b(a+2a')}{a+a'}$$

and
$$\frac{\frac{1}{2}ab \frac{a+a'}{3} + \frac{1}{2}a'b \frac{a'}{3}}{\frac{1}{2}ab + \frac{1}{2}a'b} = \frac{1}{3} \frac{a^2 + aa' + a'^2}{a+a'}.$$

Second solution. Treat the plate as a remainder from a rectangular plate.

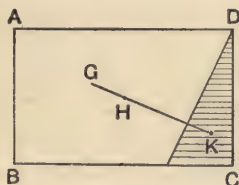


FIG. 182.

The areas of the rectangle, part cut off and remainder, are respectively

$$ab, \quad \frac{1}{2}(a-a')b, \quad \frac{1}{2}(a+a')b.$$

Hence, if we call H, K, G the centres of mass of these, and if the distances of G from the sides a and b are y and x , we can say that masses

$$\frac{1}{2}(a-a')b \text{ at K, whose coordinates are } \frac{2b}{3} \text{ and } \frac{2a+a'}{3},$$

and $\frac{1}{2}(a+a')b$ „ G, „ „ „ „ y „ x ,

have a C.M. at H, whose coordinates are $\frac{1}{2}b$ and $\frac{1}{2}a$.

$$\begin{aligned}
 \therefore \frac{1}{2}b &= \frac{\frac{1}{2}(a-a')b \frac{2b}{3} + \frac{1}{2}(a+a')by}{\frac{1}{2}(a-a')b + \frac{1}{2}(a+a')b} \\
 &= \frac{(a-a')\frac{2b}{3} + (a+a')y}{2a}; \\
 \therefore (a+a')y &= ab - (a-a')\frac{2b}{3} = \frac{ab + 2a'b}{3}, \\
 y &= \frac{b(a+2a')}{3(a+a')}.
 \end{aligned}$$

Similarly x can be found.

Third solution. We can divide the quadrilateral into triangles as in the first method, and then replace each triangle by proper masses at the vertices.

Thus the triangle ABC can be replaced by masses proportional to its area, each say $\frac{1}{2}a'b$ at A, B and C, and ACD will be replaced by $\frac{1}{2}ab$ at A, C, D, giving in all

$\frac{1}{2}(a'b + ab)$ at A and C,

$\frac{1}{2}a'b$ at B,

$\frac{1}{2}ab$ at D,

and the C.M. of the quadrilateral is the C.M. of these four particles.

Fourth solution. The quadrilateral may, of course, be conveniently divided into a rectangle and triangle, and we would continue as in the first solution. We leave the working out to the student.

3. An equilateral triangle and a square have a side in common. Find the distance of the centre of mass of the two from the common side.

4. A metal square (the shape of a capital L) is of uniform thickness, and the outer edges are of lengths 10 and 8 ins. and the breadth 2 ins. throughout. Find the distances of the C.M. from the outer edges of the square.

5. A triangle whose sides are 5, 12, 13 ins. has a quarter cut off it by a line parallel to the shortest side, and bisecting the others. Find the distances of the C.M. of the remainder from the sides including the right angle.

6. Find the centre of mass of three equal rods each of length $2a$ and forming three sides of a square.

7. Find the centre of mass of five equal rods each of length $2a$ forming five sides of a regular hexagon.

8. ABCDEF is a hexagonal plate whose centre is O ; find the C.M. of the remainder after

- (i) the triangle ABC has been removed,
- (ii) the triangle OAB has been removed.

9. Find the distance of the C.M. of a regular tetrahedron of edge $2a$ from its base.

10. From a regular tetrahedron of edge $2a$ a regular tetrahedron of edge a is cut off by a plane parallel to one face ; find the distance of the C.M. of the remainder from the base.

11. A cylindrical shell of thin sheet metal has a base of the same thickness but no top. If its radius is r and height h , find the C.M. of the whole.

12. Find the position of the C.M. of a framework of six equal bars of length $2a$ forming the edges of a regular tetrahedron.

13. Find the C.M. of a regular tetrahedron formed of thin sheet metal.

14. Find the centre of mass of a hollow cone with a base, all formed of uniform sheet metal.

15. One end of a rectangular plate is cut off obliquely, leaving a quadrilateral whose parallel sides are of lengths 6 and 4 ins. and breadth 3 ins. Find the distances of the C.M. from the square end and from the parallel sides.

16. In a cubical block a cylindrical hole is drilled perpendicular to one face, the centre of one end of the cylinder being at the centre of the cube. Find the C.M. of the remainder if the edge of the cube is $2a$ and the radius of the cylinder r .

17. In a circular sheet of metal of radius 10 ins., a circular hole of radius 3 ins. is punched whose centre is 5 ins. from the centre of the sheet. Find the position of the C.M. of the remainder.

18. Masses of 1, 2, 3, 4 lbs. are placed at the corners of a square plate of mass 6 lbs. Find the centre of mass of the whole.

19. Masses of 1, 4, 3, 5 lbs. are placed at the corners of a square plate of mass 6 lbs. Find the distance of the centre of mass of the whole from the centre of the plate.

20. Masses 1, 2, 3, 4, 5, 6 are placed at equal intervals round the circumference of a circular plate weighing 4 lbs. Find the C.M. of the whole.

21. Two angles A, C of a quadrilateral are at distances h_1, h_2 from the diagonal BD. Shew that the C.M. is at a distance $\frac{h_1 + h_2}{3}$ from BD.

22. The diagonals of a quadrilateral ABCD intersect at right angles at O. Prove that the distance of the C.M. of the quadrilateral from O is $\frac{1}{2}\sqrt{(OD - OB)^2 + (OA - OC)^2}$.

23. A pyramid with a square base is formed by piling up equal balls, there being n^2 in the base. Shew that the height of the C.M. of the pyramid above the centres of the balls forming the base is to the height of the top ball as $n : 4n + 2$.

24. Shew that the C.M. of a thin triangular frame is the same as that of particles of masses $b+c$, $c+a$, $a+b$, at the vertices of the triangle, and deduce that the C.M. is at the centre of the inscribed circle.

25. Shew how to find particles which, when placed at the vertices of a tetrahedral framework will have the same C.M. as the framework.

The edges of a tetrahedral framework are $AB = AC = AD = 13$ ins., $BC = 10$, $CD = 6$, $DB = 8$ ins. Find the distance of the C.M. from the triangle BCD.

108. Centre of Mass of any Pyramid.

Let Fig. 183 represent a pyramid on any rectilinear base ABCDE and having the vertex at O. Let H be the centre of mass of the base, and h the height of the pyramid.

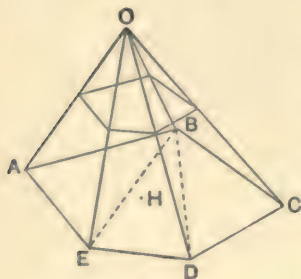


FIG. 183.

By dividing the base into triangles such as ABE, BED, CBD we can divide the pyramid into tetrahedra such as ABEO, etc.

Now the centre of mass of each of these tetrahedra is at a height $\frac{h}{4}$ above the base ;

$\therefore$ the C.M. of the whole pyramid is at a height $\frac{h}{4}$ above the base.

Also a plane drawn parallel to the base will cut the pyramid in a rectilinear area similar to ABCDE, and hence, by drawing such planes close to one another, we get a series of slices similar to the base, and therefore having the C.M. of the slice

similarly placed to that of the base. Just as in the tetrahedron it will follow that the C.M.'s of all the slices are on the line joining the vertex to the C.M. of the base, and consequently that the C.M. of the whole tetrahedron is on this line. And as the distance of the C.M. from the base is a quarter of the height of the pyramid, the C.M. will be a quarter of the way along the line AH from H, or three-quarters of the way along from A.

The above applies to any form of pyramid, including any cone (which is a pyramid standing on a curvilinear base), and in particular to the right circular cone.

Hence the **C.M.** of a **Right Circular Cone** is on its axis, and three-quarters of the way from the vertex to the base.

109. Truncated Cone or Pyramid.

If the vertex of a cone is cut off by a plane parallel to its base, the C.M. of the remainder can be found by the method of Art. 105, combined with the result of Art. 108, or may be worked out independently by integration.

110. Centres of Mass by the Calculus.

Many questions of calculating the position of the centre of mass of a body of given shape require the use of the Integral Calculus, and a few of the simpler cases will be worked out here by it. Some of them may be obtained by algebraic methods, which are, however, usually more complicated.

In the following work, ρ will be used for the density of a body, that is, its mass per unit volume, as grammes per cubic centimetre, or lbs. per cubic ft.; σ will be used for the mass per unit area of a thin plate, as grammes per square centimetre, or lbs. per square ft., of a sheet of metal.

111. Segment of a Parabola (or thin parabolic plate) cut off by an ordinate perpendicular to the axis.

Let CAC' be the segment, and let the equation of the parabola be $y^2 = 4ax$, and let the height AB of the segment be h .

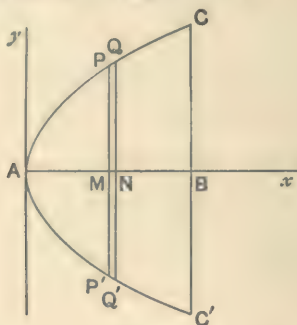


FIG. 184.

The C.M. evidently lies on AB .

Divide the parabola into strips like $PP'Q'Q$.

The length of the strip is $2y$, its breadth is dx .

$\therefore$ its area $= 2y \, dx$;

$\therefore$ its mass $= \sigma \cdot 2y \, dx$.

Now by the general formula $\frac{\sum mx}{\sum m}$ of Art. 107, in order to get the x coordinate of the C.M., we have to multiply the mass of each little element into which we divide the body by its x coordinate, add together all such products, and divide by the total mass. As the summation becomes simply integration from 0 to h in this example, we get

$$\begin{aligned} \bar{x} &= \frac{\int_0^h \sigma \cdot 2y \, dx \cdot x}{\int_0^h \sigma \cdot 2y \, dx} = \frac{2\sigma \int \sqrt{4ax} \cdot x \, dx}{2\sigma \int \sqrt{4ax} \cdot dx} = \frac{2a^{\frac{1}{2}} \int x^{\frac{3}{2}} \, dx}{2a^{\frac{1}{2}} \int x^{\frac{1}{2}} \, dx} \\ &= \frac{\left[\frac{2}{5} x^{\frac{5}{2}} \right]_0^h}{\left[\frac{2}{3} x^{\frac{3}{2}} \right]_0^h} = \frac{3}{5} \frac{h^{\frac{5}{2}}}{h^{\frac{3}{2}}} = \frac{3}{5} h. \end{aligned}$$

112. Solid Hemisphere.

Let Ox be the radius of the hemisphere perpendicular to its base.

Divide it into thin plates by planes parallel to the base. These plates are circular disks like $PP'Q'Q$ in the figure.

The area of the disk $= \pi y^2$,

thickness „ $= dx$;

$\therefore$ volume „ $= \pi y^2 dx$;

$\therefore$ mass „ $= \rho \pi y^2 dx$
 $= \rho \pi (a^2 - x^2) dx$,

since $x^2 + y^2 = a^2$;

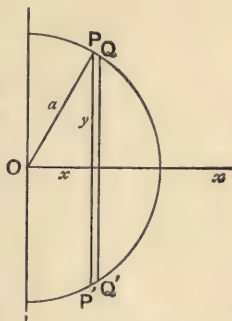


FIG. 185.

$$\begin{aligned} \therefore \bar{x} &= \frac{\int_0^a \rho \pi (a^2 - x^2) dx \cdot x}{\int_0^a \rho \pi (a^2 - x^2) dx} = \frac{\rho \pi \int_0^a (a^2 x - x^3) dx}{\rho \pi \int_0^a (a^2 - x^2) dx} \\ &= \frac{\left[\frac{a^2 x^2}{2} - \frac{x^4}{4} \right]_0^a}{\left[a^2 x - \frac{x^3}{3} \right]_0^a} = \frac{\frac{a^4}{4}}{\frac{2a^3}{3}} = \frac{3}{8} a. \end{aligned}$$

113. Zone of Sphere, that is, a portion of the surface cut off by two parallel planes.

If we draw planes as in the last example and take a narrow belt $PP'Q'Q$,

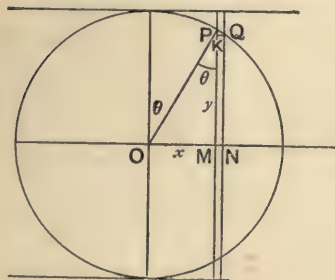


FIG. 186.

B. E. S.

M

its circumference

$$= 2\pi y = 2\pi a \cos \theta,$$

where θ = the angle OP makes with Oy ,

the breadth of the belt

$$= PQ$$

$$= QK \sec \theta$$

$$= dx \sec \theta;$$

$$\therefore \text{the area of the belt} = 2\pi a \cos \theta \cdot dx \sec \theta \\ = 2\pi a dx.$$

Now suppose a cylinder drawn just enclosing the sphere and with its axis parallel to Ox . This is called a circumscribing cylinder of the sphere. The area cut off by the same planes on this cylinder is also $2\pi a dx$.

Thus the area of any belt of the sphere is equal to the area cut off on the circumscribing cylinder between the same two planes (Archimedes's result), and the centre of mass of the belt is given by

$$\bar{x} = \frac{\Sigma 2\pi a dx \cdot x}{\Sigma 2\pi a dx},$$

and is the same as the C.M. of the corresponding belt of the cylinder, and is half-way between the bounding planes.

114. Circular Arc (or thin wire in the form of an arc of a circle).

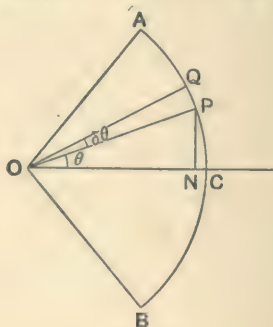


FIG. 187.

Let ACB be the wire, O being the centre of the arc, and OC the radius bisecting the angle AOB , and

let $AOC = \alpha$ in circular measure,

PQ be an element of the wire.

$$\angle POC = \theta,$$

$$\angle POQ = d\theta;$$

$$\therefore \text{arc } PQ = a d\theta.$$

Let λ = mass per unit length ;

$\therefore$ mass of $PQ = \lambda a d\theta$,

$x = ON = a \cos \theta$;

$$\therefore \bar{x} = \frac{\int_{-\alpha}^{\alpha} \lambda a d\theta \cdot a \cos \theta}{\int_{-\alpha}^{\alpha} \lambda a d\theta} = a \frac{\left[\sin \theta \right]_{-\alpha}^{\alpha}}{\left[\theta \right]_{-\alpha}^{\alpha}} = \frac{a \sin \alpha}{\alpha}.$$

For a semi-circular arc $\alpha = \frac{\pi}{2}$;

$$\therefore \bar{x} = \frac{2a}{\pi}.$$

115. Sector of a Circle.

The sector may be divided into a number of thin triangles. The C.M. of any one of these is two-thirds of the way from the centre to the circumference of the circle.

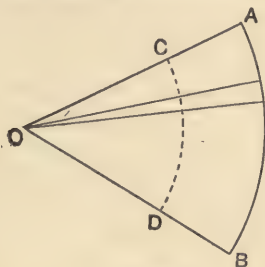


FIG. 188.

Hence the C.M. will be the same as the C.M. of the arc CD of two-thirds of the radius of AB, and hence it is at a distance from O

$$= \frac{2}{3} \cdot \frac{a \sin \alpha}{\alpha}$$

if $AOB = 2\alpha$.

Hence the C.M. of a semi-circular area is at a distance from the centre

$$\frac{2}{3} \frac{a}{\pi} = \frac{4a}{3\pi}.$$

116. Pappus's Theorems.

The following theorems are important :

(i) If a plane curve (closed or unclosed) revolves about a line in its plane, the area of the surface generated is equal to the length of the curve multiplied by the length of the path traced out by the centre of mass of the curve.

(ii) If a plane closed curve revolves about a line in its plane which does not intersect it, the volume generated by the curve is equal to the area of the curve multiplied by the length of the path of the centre of mass of the area.

(i) Let the curve AB revolve about the axis of y .

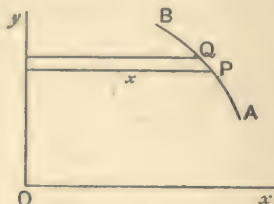


FIG. 189.

The x coordinate of the C.M. of the arc is given by

$$\bar{x} = \frac{\sum x ds}{\sum ds},$$

where ds is the length of an element PQ.

Now the area generated in a revolution by the element PQ is a belt of radius x and breadth ds , and consequently

$$\text{its area} = 2\pi x ds;$$

$$\therefore \text{the whole area} = \sum 2\pi x ds$$

$$= 2\pi \sum x ds$$

$$= 2\pi \bar{x} \sum ds$$

$$= 2\pi \bar{x} l,$$

if l is the total length.

Now $2\pi\bar{x}$ is the length of the path of the centre of mass, and hence the theorem is proved.

(ii) With similar notation, let

$\bar{X}$ = coordinate of the C.M. of the area of the curve.

S = total area.

dS = an element of area.

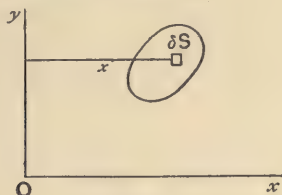


FIG. 190.

The element dS in revolving generates a ring whose volume is

$$2\pi x dS;$$

$$\therefore \text{total volume} = \Sigma 2\pi x dS;$$

but
$$\bar{X} = \frac{\Sigma x dS}{\Sigma dS};$$

$$\therefore \text{volume} = 2\pi \bar{X} \Sigma dS$$

$$= 2\pi \bar{X} S$$

= total area $\times$ length of path of the C.M.
of the area.

117. Pappus's Theorems may be used in some cases for finding the centre of mass of a line or area.

Thus, if the curve is a semi-circle rotating about its base, the area generated is the area of a sphere and $= 4\pi a^2$.

The length of the curve $= \pi a$;

$$\therefore 4\pi a^2 = 2\pi \bar{x} \times \pi a;$$

$$\therefore \bar{x} = \frac{2a}{\pi} \text{ (cf. Art. 114).}$$

Also the volume generated = the volume of the sphere $= \frac{4}{3}\pi a^3$.

The area of the semi-circle = $\frac{1}{2} \pi a^2$

$$\therefore \frac{1}{3} \pi a^3 = 2\pi \bar{X} \cdot \frac{1}{2} \pi a^2,$$

$$\bar{X} = \frac{4a}{3\pi} \text{ (cf. Art. 115).}$$

Examples. 1. In a cubical block of edge $2a$ a hemispherical cavity of radius r is hollowed out with its centre at the centre of one of the faces. Find the C.M. of the remainder.

2. In a cubical block of edge $2a$ a hole is drilled consisting of a cylinder of radius r terminated by a cone of height h . If the vertex of the cone is at the centre of the block, find the C.M. of the remainder.

3. Find the C.M. of a truncated right circular cone, the radii of whose ends are a and b and height h .

Find also the C.M. of

(i) the curved surface of the truncated cone,

(ii) the whole surface.

4. A log of wood 12 ft. long is a truncated cone whose ends are of radii 2 and 3 ft. respectively. Find the volume and C.M. of the whole.

5. With the result of Ex. 2, Art. 107, verify Pappus's theorem by finding the volume of a truncated cone by revolving the area in that example about the side b .

6. Shew that the C.M. of a homogeneous solid pyramid whose base is a parallelogram is the same as that of four equal particles at the angles of the base with a particle of four-thirds of the mass of each at the vertex.

7. Find the C.M. of a segment of a circle.

8. Find the C.M. of a paraboloid of revolution (the solid formed by revolving a parabola about its axis) bounded by a plane perpendicular to its axis.

9. Find the C.M. of the portion of a sphere bounded by two parallel planes.

10. Find the volume and centre of mass of a segment of a sphere.

11. A cylindrical hole of radius r is cut centrally and perpendicularly to the base through a hemisphere of radius a . Shew that the C.M. of the remainder is at a distance $\frac{3}{8}\sqrt{a^2 - r^2}$ from the base.

12. Two faces of a solid bounded by planes are rectangles of sides a_1, b_1 ; a_2, b_2 , the sides a_1, a_2 being parallel. The distance between these faces is h . Shew that the distance of the C.M. from the rectangle a_2, b_2 is
$$\frac{a_2 b_2 + 3a_1 b_1 + a_2 b_1 + a_1 b_2}{2a_2 b_2 + 2a_1 b_1 + a_2 b_1 + a_1 b_2} \cdot \frac{h}{2}.$$

13. Shew that if a slice of a spherical shell of any thickness is made by two parallel planes so as to form a ring, the C.M. is midway between the parallel planes.

14. A uniform thin wire forms a coil of radius a and consists of m complete turns and $\frac{1}{n}$ of a turn. Shew that its C.M. is at a distance $an \sin \frac{\pi}{n} / (mn + 1)\pi$ from the centre.

15. A body is bounded by a rectangle of sides $2a$ and $2b$ ($b < a$) and four planes through these at equal inclinations θ to the plane of the rectangle. Shew that the distance of the C.M. from the rectangle is

$$\frac{b(2a-b)}{2(3a-b)} \tan \theta.$$

16. Find the area of the parabola $y = \frac{x^2}{4a}$ between the axis of x , the curve, and the ordinate $x = 4a$, and find the coordinates of the C.M. of this area.

If this area revolves about the axis of x , find the volume and C.M. of the solid generated.

118. Upsetting of a Body on a Plane.

If a body is placed gently on a horizontal plane and left to itself, it may remain as placed or it may fall over. Think of it as held for a moment. It is in contact with the plane at certain points or over certain areas, and a closed convex polygon may be drawn to just enclose all these areas, and this polygon may be called the base of the body. A convenient way of thinking of this base as drawn is to suppose a string drawn as tight as possible and enclosing all the contacts. Thus, if a table stands on four legs a string passed tightly round the four feet will enclose the area, which is called the base. The condition that when left to itself the body will remain standing is that the vertical through the centre of mass may pass through

the base. For the forces acting are the weight vertically downwards, and the reactions at the contacts acting vertically upwards. The reactions can be compounded into a single vertical force which must act within the base, and if the centre of mass is somewhere above the base, these reactions will be able to adjust themselves so as to balance the weight. But if the vertical through the centre of mass falls outside the base the resultant reaction cannot possibly act in the line of the weight, and the body will upset when left to itself.

The same applies to a body on an inclined plane when there is sufficient friction to prevent it from sliding. Thus, if a rectangular block is placed on a plane which is then tilted up gradually, the block may topple over without sliding, or may slide without toppling, according to the dimensions of the body and the coefficient of friction.



FIG. 191.

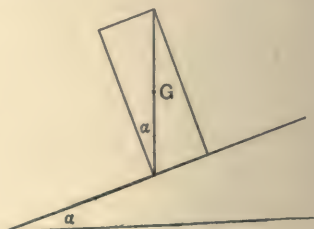


FIG. 192.

In Fig. 191 let $2h$ be the height,

$2a$ the breadth of the base.

If the body is still at rest when the plane makes an angle α with the horizontal, and if R , F are the normal reaction and friction,

$$R = W \cos \alpha,$$

$$F = W \sin \alpha.$$

For it to be on the point of sliding,

$$\frac{F}{R} = \mu = \tan \alpha.$$

On the other hand, for it to be on the point of toppling, G must be vertically above A, as in Fig. 192.

$$\tan \alpha = \frac{a}{h}.$$

Hence, as the plane is tilted up, the block will slide before toppling if

$$\mu < \frac{a}{h},$$

and will topple before sliding if

$$\mu > \frac{a}{h}.$$

EXAMPLES.

1. A circular table of weight W and radius $2a$ stands on four vertical legs at a distance a from the centre. What is the least weight that can upset the table by being placed on it?

2. A number of equal bricks are piled on top of one another, the lowest being on a horizontal table, and each successive one projecting over the one below it a distance equal to $\frac{1}{n}$ of the length of a brick. How many can be thus placed without their upsetting?

3. A right circular cone, whose base has a radius of 1 ft. and whose height is 4 ft., is placed on a plane which is then tilted up. If the coefficient of friction is greater than unity, shew that the cone will topple before sliding, and find the inclination of the plane when toppling takes place.

4. A circular cylindrical cup with a flat bottom is made of sheet metal, of weight w per unit area. It stands on a horizontal table, and a rod of weight W is placed in it and projects over the edge. Shew that the whole will upset if the length of the rod is greater than

$$\sqrt{4r^2 + h^2} \left\{ 2 + \frac{w}{W} (\pi r^2 + 2\pi r h) \right\},$$

where h is the height of the cylinder and r its radius.

5. A number of strings knotted together at a common point O pass over the smooth edge of a horizontal circular table on which O lies, and carry masses hanging freely at the ends. Shew that if the system is in equilibrium, either the centre of mass of the masses is vertically below O or all the strings lie in the same vertical plane.

6. A sphere of radius r , whose centre of mass is at a distance a from the centre, rests on two rough planes each making an angle 45° with the horizontal. Shew that when slipping is about to take place the line joining the centre of mass to the centre of the sphere makes an angle θ with the vertical, where

$$\sqrt{2}a \sin \theta = r \sin 2\lambda,$$

where λ is the angle of friction.

7. A rectangular block of weight W , and of edges $2a$, $2b$, $2c$, stands on a rough horizontal plane. What horizontal force applied at the middle of the upper edge will be required to overturn it if the edges $2a$ are vertical and the force is parallel to the edges $2b$?

Find also the work necessary to be done to upset it.

8. A ladder has its centre of mass one-third of the way up. It rests against a vertical wall and on a horizontal plane. Shew that if the coefficient of friction is $1/3$ at each end, it is in limiting equilibrium when the inclination to the horizontal is $\tan^{-1} \frac{7}{9}$.

CHAPTER XI.

VIRTUAL WORK.

119. We have already explained the Principle of Virtual Work as applied to forces acting on a particle. We must now extend it to the case of any forces acting on a rigid body.

It is first necessary to examine the displacements possible for a rigid body capable of moving in one plane. A fuller treatment will be found in my *Elementary Dynamics*, Arts. 168-174.

From the meaning of a rigid body any two points of the body remain at the same distance from one another, however the body is moved about.

A body is said to be **moving in one plane** when a plane fixed in the body moves in a plane fixed in space. Different points of the body then move in parallel planes. For example, if a cube slides about on a horizontal table with the same face always in contact with the table, it is said to be moving in one plane. In this case all points of the cube move in horizontal planes.

We will only deal here with motion in a plane.

In any such case all points in a straight line perpendicular to the planes of motion undergo the same displacement in the same interval of time, and therefore the displacements of the whole body may be represented by the displacements of all points in any one of these planes.

120. There are two fundamental kinds of displacement, **Translation and Rotation.**

In Translation all points of the body move equal and parallel distances; in other words, all straight lines in the body move into new positions parallel to their old positions.

In Rotation one point (or rather line perpendicular to the planes of motion) is fixed, and the whole body turns about this line or axis.

Any displacement whatever of a rigid body in a plane may be thought of as made up of a translation and a rotation.

For firstly, if two points A and B in a plane of motion come to new positions A' and B', A'B' being equal to AB, the displacement may be thought of as carried out in two steps, first by a

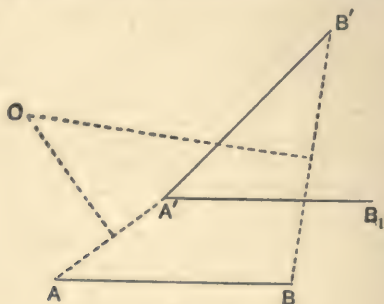


FIG. 193.

translation bringing A to A', and B to a point B₁, and then a rotation about A' bringing the line A'B₁ into the position A'B', and when the two points are fixed in their final positions every other point is fixed.

The displacement can also be produced by a rotation about a properly chosen point. For if AA' and BB' are bisected at right angles, and the bisectors meet in O, the triangles OAB, OA'B' are congruent, and therefore O is the same point of the body in both positions, and as A can be brought into its new

position by a rotation about O, all points of the body are brought into their new positions by the same rotation at the same time.

In applications to Virtual Work in Statics we only require to deal with infinitesimal displacements, and we will generally calculate the infinitesimal displacements by means of the differential calculus.

121. We will prove now that if a body is at rest under the action of any forces and the body is displaced a small amount in any way, the forces still acting at the same points and in the same directions as before, the total work done by and against the forces is zero, provided the squares of the small displacements are neglected; or expressed in other words, the work done is zero to the first order of small quantities.

Take first the case of translation alone, and suppose the body to be moved with translation δs in any direction. If a force P on the body makes an angle α with the displacement, the work done by P is

$$P \cos \alpha \cdot \delta s,$$

and the total work done by all the forces is

$$\begin{aligned} & \Sigma P \cos \alpha \cdot \delta s \\ &= \delta s \Sigma P \cos \alpha \\ &= 0, \end{aligned}$$

since $\Sigma P \cos \alpha = 0$, because the forces are in equilibrium and therefore the sum of the resolved parts in any direction is zero.

Next suppose the body to rotate through any angle $\delta \theta$ about any point O, and suppose a point A comes in consequence to A'. Take any line through O as axis of x , and suppose OA makes an angle α with Ox .

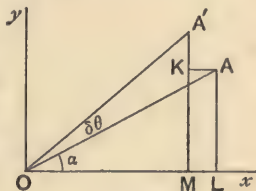


FIG. 194.

Let $OA = r$.

Then the displacement of A can be represented by the increases δx and δy in the coordinates.

$$\begin{aligned}\text{Now } \delta x &= r \cos (\alpha + \delta \theta) - r \cos \alpha \\ &= r (\cos \alpha \cos \delta \theta - \sin \alpha \sin \delta \theta) - r \cos \alpha ;\end{aligned}$$

also, omitting squares of $\delta \theta$ (see *Elementary Dynamics*, Art. 113),

$$\cos \delta \theta = 1,$$

$$\sin \delta \theta = \delta \theta,$$

$\delta \theta$ being in circular measure ;

$$\therefore \delta x = -r \sin \alpha \cdot \delta \theta = -y \delta \theta.$$

$$\begin{aligned}\text{Again, } \delta y &= r \sin (\alpha + \delta \theta) - r \sin \alpha \\ &= r (\sin \alpha \cos \delta \theta + \cos \alpha \sin \delta \theta) - r \sin \alpha \\ &= r (\sin \alpha + \cos \alpha \cdot \delta \theta) - r \sin \alpha \\ &= r \cos \alpha \cdot \delta \theta \\ &= x \delta \theta.\end{aligned}$$

Now, if P is the force acting at A , and its components are X and Y , we have proved (Art. 42) that the work done by P is the sum of the works done by X and Y , also that the work done by X in the displacement AA' is the sum of the works done in the displacements δx and δy , of which AA' is the resultant.

$$\begin{aligned}\therefore \text{the work done} &= X \delta x + Y \delta y \\ &= (Yx - Xy) \delta \theta.\end{aligned}$$

Hence the total work done by all the forces is

$$\begin{aligned}&\Sigma (Yx - Xy) \delta \theta \\ &= \delta \theta \Sigma (Yx - Xy) \\ &= 0,\end{aligned}$$

since $\Sigma (Yx - Xy) = 0$ for equilibrium.

122. We could have shortened the above work considerably if we had used calculus methods, and simply put that, since

$$x = r \cos \alpha,$$

$$\delta x = \frac{dx}{d\alpha} \delta \alpha = -r \sin \alpha \cdot \delta \alpha, \quad \text{and} \quad -y \delta \alpha$$

where $\delta \alpha$ has been used for the rotation instead of $\delta \theta$.

Similarly, since

$$y = r \sin \alpha,$$

$$\delta y = r \cos \alpha \cdot \delta \alpha = x \delta \alpha$$

123. As the total virtual work done in any small displacement of a rigid body is zero, it is evident that the same is true of any number of connected bodies. That is to say, if any displacements are given to the bodies with the understanding that the forces remain constant in magnitude and direction, the virtual work of the forces is zero, to the first order of small quantities.

The displacements may be any whatever, but in general it is advisable to take displacements consistent with the geometrical relations of the bodies. For example, two rods pressing against one another, or jointed together, will be given displacements in which they remain in contact. If this is done it is clear that the reactions between the bodies will not occur in the equation of virtual work. For the work done by the force acting on the one rod is equal and opposite to the work done by the reaction acting on the other. Similarly, the tension of an inextensible string will not appear, as has been pointed out in Chapter IV. (Art. 46), but a displacement may be chosen which produces a change in the length of the string, and then the virtual work of the tension will appear, and will be equal to the tension multiplied by the diminution of length. Such a displacement, it will be noticed, may be made, though not consistent with the fact that the string is inextensible, because the displacements are only virtual, that is to say, "imagined."

124. We will work a few very easy examples, which can also be easily worked by the methods of the former chapters, but they will illustrate the method of virtual work, and shew how, by this method, we can often calculate the force we want to find without bringing all the other forces into the equations.

The student must be careful about the sign of the work done in a displacement, remembering that it is positive if the resolved part of the displacement along the line of action of the force is in the direction of the force, and if in the opposite direction negative.

Examples. 1. A rod AB of weight W is supported at A by a smooth hinge, and a string passes from B over a pulley C, vertically above A, and supports a weight P . If $AB = AC$, find the angle CAB in equilibrium.

Let $CAB = \theta$.

Make a small change (increase) $\delta\theta$ in θ .

We will have to find the work done by W and P . The reaction at A does no work.

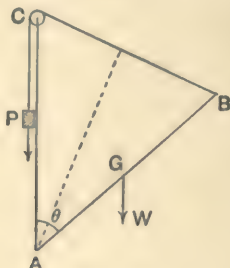


FIG. 195.

Now, if $AB = 2a$, and G is the middle point of AB , the height of G above $A = a \cos \theta$;

$$\therefore \text{change in height of } G = -a \sin \theta \cdot \delta\theta;$$

$$\therefore \text{work done by } W = Wa \sin \theta \cdot \delta\theta;$$

also

$$\text{the length of } BC = 4a \sin \frac{\theta}{2};$$

$$\therefore \text{change in } BC = 2a \cos \frac{\theta}{2} \cdot \delta\theta;$$

$$\therefore P \text{ ascends a distance} = 2a \cos \frac{\theta}{2} \cdot \delta\theta;$$

$$\therefore \text{work done by } P = -2Pa \cos \frac{\theta}{2} \cdot \delta\theta;$$

$$\therefore Wa \sin \theta \delta\theta - 2Pa \cos \frac{\theta}{2} \delta\theta = 0,$$

$$W \sin \theta = 2P \cos \frac{\theta}{2},$$

$$\therefore \text{either} \quad \cos \frac{\theta}{2} = 0, \text{ giving } \theta = 180^\circ$$

$$\text{or} \quad \sin \frac{\theta}{2} = P/W.$$

2. Two rods AB, AC, of weights W, W' , and of equal length $2a$, are smoothly jointed at A and stand in a vertical plane with the ends B, C on a smooth table and connected by a string. If they make angles α with the horizontal, find the tension of the string.

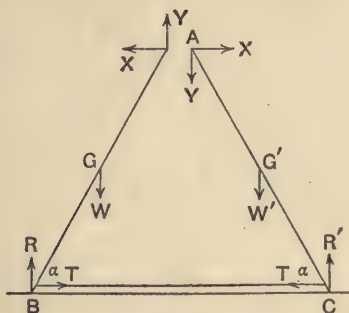


FIG. 196.

The forces are as in Fig. 196. Give a displacement by increasing α . The original height of G and G' above the table is $a \sin \alpha$;

$$\therefore \text{increase in height} = a \cos \alpha \delta \alpha;$$

$$\therefore \text{work done by } W \text{ and } W' \text{ is } -(W + W')a \cos \alpha \cdot \delta \alpha.$$

$$\text{The length of the string} = 4a \cos \alpha;$$

$$\therefore \text{increase in length} = -4a \sin \alpha \delta \alpha,$$

$$\text{work done by the tension} = T 4a \sin \alpha \delta \alpha$$

$$-(W + W')a \cos \alpha \delta \alpha + T 4a \sin \alpha \delta \alpha = 0;$$

$$\therefore T = \frac{1}{4}(W + W') \cot \alpha.$$

It will be noticed how none of the forces X, X', R, R' come into the equation. If there were friction at B and C it would appear in the equation of virtual work, and so also would a friction couple at the joint. Thus, if it takes a couple L to move the joint there would be work done by friction in the displacement of $-L \cdot 2\delta\alpha$, since the angle between the rods is changed by $2\delta\alpha$, and the work done by a couple is the product of the couple into the angle through which the arm is twisted.

It will also be noticed that the work done by each force is obtained in terms of a single displacement. This is always necessary where the position of the system is determined by a single coordinate such as the θ in Question 1 or the α in Question 2. Cases involving two independent coordinates will not be dealt with in this book.

125. The method of virtual work may be applied to just rigid frames, and can be used to calculate any stress without bringing the rest into the equation.

Ex. 3. Take the framework worked out in Art. 80, and suppose we wanted to find T directly by virtual work. Imagine the bar BC removed, but the forces T to be applied in some way to the joints. This will still keep the frame in equilibrium, but it is deformable, and we can imagine the two halves to rotate each through an angle $\delta\theta$ about A and D . This will be found to cause B and C to descend distances $a\delta\theta$, while the distance BC will be shortened by an amount $2a\sqrt{3}\delta\theta$. Hence the equation of virtual work is

$$-T2a\sqrt{3}\delta\theta + 10a\delta\theta + 20a\delta\theta = 0,$$

$$T2\sqrt{3} = 30,$$

$$T = 5\sqrt{3}.$$

126. Stable, Unstable, and Neutral Equilibrium.

If a thin cylinder, such as a pencil, is placed carefully with one end on a horizontal plane, it may stand upright; but if one tries to stand an equally thin stick, three feet long, upright on a table, he will probably fail, or at any rate it will not stand up for more than a fraction of a second. The upright position is, however, one of equilibrium, but the slightest shake, an almost infinitesimal puff of air, is sufficient to upset it.

If the same stick is supported at the upper end and allowed to hang down, it may be disturbed from that position, but it returns to it again after the disturbance.

These two cases are called Unstable and Stable equilibrium respectively. We may define the terms by saying that if a body is in equilibrium, and on being displaced slightly from its position of equilibrium it returns towards that position again, it is said to be in stable equilibrium; if, however, it moves further away from the position of equilibrium, it was in unstable equilibrium.

In the examples of the pencil and stick above, the body is in stable equilibrium when the centre of mass is as low as it can get, and in unstable equilibrium when it is as high as it can get.

If a sphere rolls in a vertical plane along a pair of rails, it can rest in stable equilibrium at a point of minimum altitude, and could be in unstable equilibrium at a point of maximum altitude. If a body can move without the height of the centre of mass changing, it is said to be in Neutral equilibrium, for on displacement it will neither tend to move further away nor to return. Such is the case when a sphere is placed on a horizontal plane.

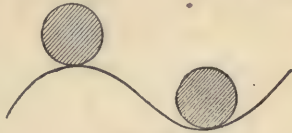


FIG. 197.

Good illustrations of the different kinds of equilibrium may be got from bodies floating on water.

Thus, an ordinary block of wood, if floating in water, is in stable equilibrium as far as vertical displacements are concerned, for if pushed downwards and released it oscillates, and finally returns to its old position, and the same also if it is raised a little. But if moved horizontally there is no tendency for it either to go further or return; it is therefore in neutral equilibrium for horizontal displacements, and may even be in unstable equilibrium at the same time for a rotational displacement about a horizontal axis.

Stable and Unstable equilibrium may generally be best treated by the method of virtual work, but to distinguish between the two we have to examine the work done, taking into account terms of the second degree in the displacement. The simplest case is where the only work done is by or against gravity acting on one body only. In this case, if there is equilibrium no work is done in a displacement when terms of the first order only are considered. In other words, the centre of mass must begin to move horizontally.

When we take terms of the second order into consideration it will be seen that if the centre of mass is lowered in a small displacement, or work is done by gravity, the equilibrium is

unstable, for the body will continue to move as long as the motion will lower the centre of mass. On the other hand, if in a small displacement the centre of mass is raised, the equilibrium is stable.

Example. Let A be a portion of a fixed sphere of radius a , B a segment of another sphere of radius b , which rests in equilibrium on the highest point of A. If the centre of mass of B is at a distance h from the point of contact, find whether the equilibrium is stable or not, the spheres being supposed so rough that the only displacement possible is a rolling of one on the other.

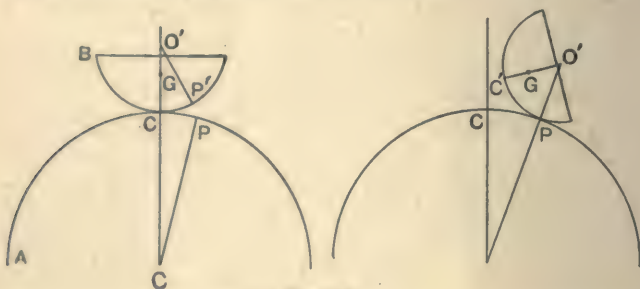


FIG. 198.

To determine whether it is unstable or stable, we must suppose the upper sphere to roll a short distance on the lower, and examine whether the centre of mass G of the upper is lowered or raised in the process. In the former case the equilibrium is unstable, in the latter stable.

Let, then, the upper sphere roll so that the point of contact is now at P, and let OP make a small angle θ with OC, O being the centre of the sphere.

If C' is the point that was in contact with C, the highest point of the fixed sphere,

$$C'P = CP.$$

Let the angle

$$C'O'P = \phi;$$

$$\therefore C'P = b\phi;$$

also

$$CP = a\theta;$$

$$\therefore b\phi = a\theta;$$

$$\therefore \phi = \frac{a}{b} \theta.$$

Consequently $O'C'$ has turned through an angle

$$\theta + \phi \text{ or } \frac{a+b}{b} \theta.$$

The new height of G above O is the projection on the vertical of the broken line $OO'G$ and

$$= (a+b) \cos \theta - (b-h) \cos (\theta + \phi).$$

The old height was

$$a+h \text{ or } (a+b) - (b-h);$$

$\therefore$ the rise of G

$$\begin{aligned} &= (a+b) (\cos \theta - 1) - (b-h) \{ \cos (\theta + \phi) - 1 \} \\ &\approx - (a+b) \frac{\theta^2}{2} + (b-h) \frac{(\theta + \phi)^2}{2} \\ &\approx - (a+b) \frac{\theta^2}{2} + (b-h) \left(\frac{a+b}{b} \right)^2 \frac{\theta^2}{2} \\ &\approx (a+b) \frac{\theta^2}{2} \left\{ \frac{(a+b)(b-h)}{b^2} - 1 \right\} \\ &\approx \frac{a+b}{b^2} \frac{\theta^2}{2} (ab - ah - bh). \end{aligned}$$

This is positive if

$$ab - ah - bh > 0$$

or

$$(a+b)h < ab$$

or

$$h < \frac{ab}{a+b}$$

or

$$\frac{1}{h} > \frac{1}{a} + \frac{1}{b}.$$

In this case equilibrium is stable.

If $\frac{1}{h} < \frac{1}{a} + \frac{1}{b}$ it is unstable.

If $\frac{1}{h} = \frac{1}{a} + \frac{1}{b}$, the equilibrium is neutral to the same degree of approximation, but on examining terms of higher degrees it will be found that it is not really neutral, but unstable.

EXAMPLES.

1. Four uniform bars, each of weight w and length a , are smoothly jointed together to form a rhombus $ABCD$. AB is fixed horizontally, and a light string joins A to C . Find the tension in the string if the angle DAB is α .

2. A smoothly jointed rhombus ABCD, of side a , is formed of uniform rods, and is supported at the point A and kept from collapsing by a light elastic string AC of natural length d and modulus λ . Shew that the length of string in the position of equilibrium is $d \left(1 + \frac{W}{2\lambda} \right)$, where W is the total weight.

3. A heavy rod rests with one end on a smooth horizontal table and the other against a smooth vertical wall. The lower end of the rod is connected with a point in the wall by a horizontal elastic string of natural length $\frac{\sqrt{2}}{3}$ of the length of the rod, and modulus equal to the weight of the rod. Shew that there is equilibrium when the rod makes an angle of 45° with the vertical.

4. AB, BC, CD are three equal heavy rods smoothly jointed at B and C. A, C, and B, D, are joined by light strings of lengths $\sqrt{3}$ times that of a rod, and the frame rests in a vertical plane with AD on a smooth horizontal floor. Find the tensions of the strings.

5. A smoothly jointed parallelogram of uniform bars ABCD has the mid points of AD, BC smoothly jointed at two fixed points in the same vertical so as to form a Roberval's Balance. To the middle of the top bar AD is fixed at right angles a pointer of length l and weight w per unit length. Shew that when weights W, W' are attached at A and D respectively, the inclination θ of the pointer to the vertical is given by

$$\cot \theta = \frac{w}{W - W'} \cdot \frac{l^2}{a},$$

where $AD = a$.

6. A rod AB of weight W is hinged at A and supported by a string attached at B and passing over a pulley at C vertically above A, and then along an arc CE of a smooth vertical circle of which A is the centre. A weight W is attached to the end of the string and can slide along the arc. If $AC = AB$, and the whole length of the string is equal to the quadrant of the circle, prove that there will be equilibrium when AD makes an angle θ with the horizontal, where

$$\theta = 2 \cos \theta,$$

θ being in circular measure.

Verify with your tables that $\theta = 59^\circ 1'$ nearly.

7. ABCD is a jointed rhombus of equal rods. AB is fixed horizontally, and a string of natural length $\frac{3}{2}AB$ and modulus equal to the weight of one bar connects AC. Shew that BC makes an angle 2θ with the horizontal given by

$$6 \cos 2\theta - 2 \sin 2\theta + 3 \sin \theta = 0.$$

8. A square formed of four uniform heavy rods, smoothly jointed at the corners, is suspended from one of the corners and kept square by a weightless rod along the horizontal diagonal. Shew that the thrust in this rod is half the weight of the square.

9. A weightless, smoothly jointed rhombus has its opposite corners joined by stretched strings. Shew that the tensions of the strings must be in the ratio of their lengths.

10. ABCD is a jointed rhombus of weightless rods with another weightless rod joining BD. Equal forces F are applied at A and C along AC. Find the stress in BD.

11. A rectangular block of wood, whose vertical dimension is a , rests on top of a fixed sphere of radius b . Is the equilibrium stable or unstable ?

12. A hemisphere of radius a rests on top of a fixed sphere of radius b . Find whether the equilibrium is stable or unstable

(i) When the curved surfaces are in contact.

(ii) When the plane surface of the hemisphere is in contact with the sphere.

13. A half cylinder of radius a rests on top of a fixed sphere of radius b . Find if the equilibrium is stable or unstable when the curved surfaces are in contact :

(i) For displacements at right angles to the length of the cylinder.

(ii) For displacements in the direction of the length.

Find also if it is stable or unstable when the flat surface is downwards.

14. A heavy rod rests horizontally on two smooth planes, each at an angle 45° with the horizontal. Is the equilibrium stable or unstable ?

15. Two equal rods, each of length $4a$, are jointed together at one end and rest in equilibrium, touching a horizontal cylinder of radius a . Prove that in equilibrium the rods make an angle 45° with the vertical. Is the equilibrium stable or unstable ?

CHAPTER XII

GRAPHICAL STATICS

127. We have used graphical methods for solving examples first in Art. 24, where the forces acted at a point, and secondly in Chap. VII., in obtaining the stresses in a weightless framework. But in the latter case it was because the question was reduced to one of dealing with the forces acting on each pin separately, or in other words to a series of cases of forces acting at a point, that we were able to find the forces graphically.

We now come to deal with the more difficult case when any forces in a plane act on a body. We will first shew how to obtain in magnitude, direction and position, the resultant of any number of forces acting in any position in a plane, and the graphical conditions for equilibrium will naturally follow.

It will be best to take first the case of three forces not meeting in a point, and when this is grasped the method can be applied at once to any number of forces.

128. Let then P, Q, R , be three forces given completely in magnitude, direction and line of action.

We have seen that the resultant is the same in magnitude and direction as if the three forces met in a point, and can therefore be determined at once by drawing a force diagram (Fig. 200) $ABCD$, with its sides representing the forces in magnitude and direction; then AD represents the resultant

in magnitude and direction. Its position remains to be determined.

Take any point O in the plane inside or outside $ABCD$, but not on its perimeter, and join OA , OB , OC , OD .

Next, in Fig. 199, take any point p on P and draw pq parallel to OB to meet Q in q .

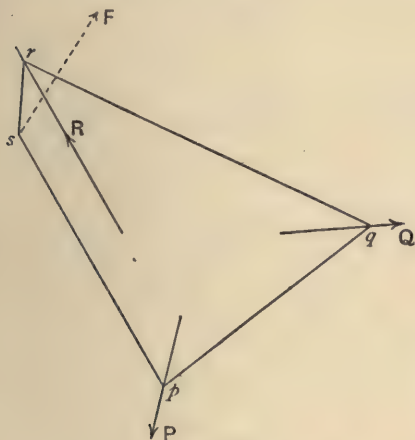


FIG. 199.

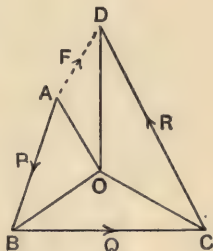


FIG. 200.

Similarly draw qr parallel to OC to meet R in r .

„ „ rs „ OD ,

„ „ ps „ OA ,

the last two lines meeting in s . Then s will be a point on the resultant.

For from Fig. 200, P is equal to the resultant of forces represented by AO and OB , and therefore P can be replaced by forces represented in magnitude by AO and OB acting along sp and qp .

Similarly Q can be represented by BO and OC acting along pq and rq ; and R can be replaced by forces represented by CO , OD , acting along qr and sr .

Suppose the force polygon is closed so that D coincides with A (Fig. 202), and therefore ps and rs being both parallel to OA are parallel to one another (in Fig. 201 rs is replaced by rs'). Hence the funicular polygon is unclosed, and it will be seen by Art. 128 that we have replaced the three forces by two, each represented by OA , acting along sp and $s'r$. The forces therefore in this case are equivalent to a couple whose moment is obtained by multiplying the force represented by OA , by the distance between ps and rs' .

If, however, ps and rs' coincide the forces are evidently in equilibrium, and the funicular polygon (for this case of three forces) is a triangle.

It will readily be seen that the same results apply to any number of forces, though, as three non-parallel forces in equilibrium necessarily meet in a point, it might at first be expected that this would be a special case. That the same result is true for four forces will, however, be seen at once by referring to Figs. 199 and 200. For if the force F is reversed we have four forces P, Q, R, F , in equilibrium, and the funicular is a quadrilateral with an angular point on each of the forces.

130. We may therefore state the general results as follows :

When n forces reduce to a single force the force polygon is open, the closing or $(n+1)^{\text{th}}$ side giving the resultant force, while the funicular has $n+1$ sides in all, the intersection of the first and last giving a point on the resultant force.

If the n forces reduce to a couple the force polygon is closed (having n sides), and the first and last sides of the funicular are parallel.

If the n forces are in equilibrium the force polygon is closed (having n sides), and the first and last sides of the funicular coincide, so that it also is a polygon of n sides.

131. Parallel Forces.

The case of parallel forces only differs from the general case in the form of the force polygon, the vertices of which lie in a straight line.

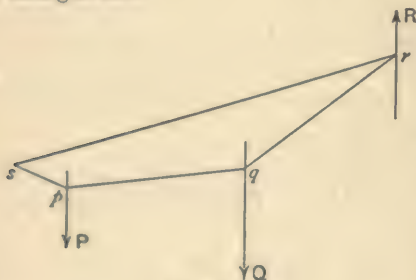


FIG. 203.

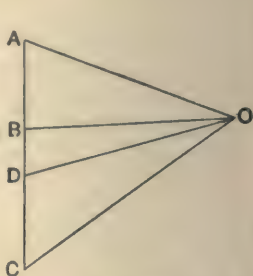


FIG. 204.

In Fig. 204 AB, BC, CD represent the forces as before, and the construction is the same, and a point s on the resultant is found as before.

If we take the simplest case of two parallel forces it will easily be seen that the method is identically the same as that by which the resultant of two parallel forces is obtained in Chapter III.

Examples. 1. Prove this statement.

2. Examine what happens to the funicular if O is taken on the closing side of the force polygon.

3. Examine what happens if the pole is taken

- (i) on a side of the force polygon,
- (ii) at a vertex of the force polygon.

4. Is it possible to begin the funicular polygon at the intersection of two forces?

132. It is easy to find the pressures exerted by a girder on its supports by means of the funicular polygon.

Thus let Fig. 205 represent a beam supported at the ends and loaded as shewn (the weight of the beam has been neglected),

and suppose we want the position of the resultant load and the pressures on the supports. Draw the force polygon (a straight

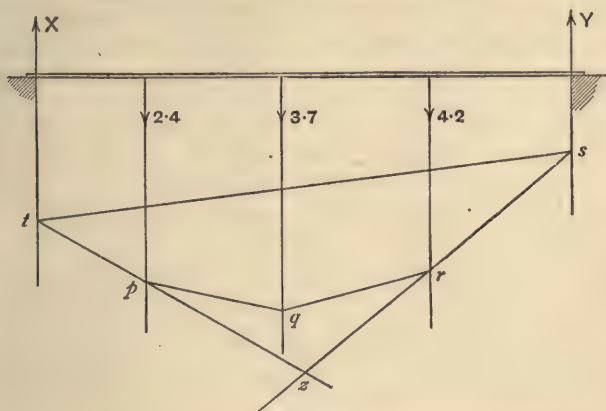


FIG. 205.

line) AB, BC, CD, representing the three loads; then DA will represent the sum of the pressures Y and X. Take the pole

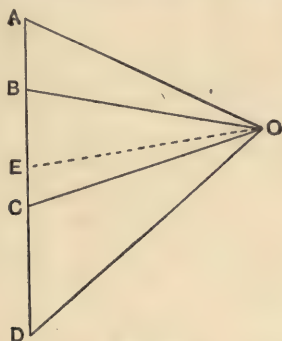


FIG. 206.

O and construct the funicular beginning at any point t , and so getting $tpqrs$. Join st , and draw OE parallel to st . Then

ABCDEA is the complete force polygon, and the reactions X and Y will be represented by EA and DE. In the example these are read off as 4.8 and 5.2. Also, if tp , sr meet in z the resultant load acts through z .

133. A few points in general remain to be noticed.

The result must be the same whatever point O is taken as *Pole* in the force diagram. Thus O may be within or without the force polygon. For different points O different points s will be obtained, but they must all lie on the line of the resultant force.

Also any point on P may be chosen as the starting point. Corresponding sides of two funiculars with different starting points p , but the same pole O , will be parallel, but the two funiculars will not necessarily be similar. Funiculars for different poles will differ more.

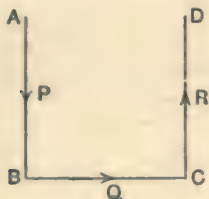


FIG. 207.

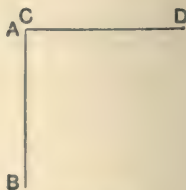


FIG. 208.

Further, it is to be observed that there is no necessary order for taking the forces when drawing the force polygon.

In choosing any particular points for the pole O and starting point p , the student may find the funicular getting off his paper, but with a little practice he will soon avoid that.

The student should draw the funiculars for many simple cases—two forces, three forces at a point, and other cases. With a given set of forces he should draw them for different positions of the pole and different starting points and different

order of forces in the force polygon, and observe if his results are consistent.

Thus take the simple case of three equal forces along three sides of a square in order. The force polygon may be the three sides, as in Fig. 207, or we may take them in the order P, R, Q, when the force polygon becomes Fig. 208.

EXAMPLES.

1. Obtain the resultant for Fig. 207 by drawing the funicular
 - (i) when the forces are taken in the order P, R, Q, and the pole is at the centre of the square ;
 - (ii) when the forces are taken in the order P, Q, R, and the pole is at the mid-point of BD in Fig. 208 ;
 - (iii) forces as (ii), and pole at corner of the square opposite A.

2. Find the position of the resultant of the parallel forces shewn in diagram 209.



FIG 209.

3. A girder, 20 ft. long and of weight 5 tons, supports loads 5.3, 8.2, 7.6 tons at points 6.8, 11.4, 15.1 ft. from one end. Find the position of the resultant weight and the pressures at the supports.

4. Find the resultant of the three forces shewn (Fig. 210), where the sides of the triangle are $a=6$, $b=4$, $c=7$.

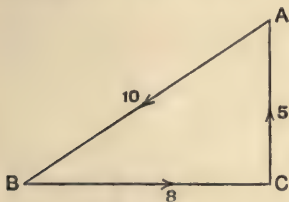


FIG. 210.

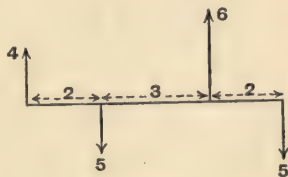


FIG. 211.

5. Shew that the parallel forces in Fig. 211 are equivalent to a couple, and find its moment.

6. Find graphically the resultant of the forces in Fig. 93, Art. 66.



FIG. 212.

7. Shew graphically that the forces in Fig. 212 are in equilibrium, the sides of the rectangle being 6 and 8 ft.

CHAPTER XIII.

FORCES IN THREE DIMENSIONS, ACTING AT A POINT.

134. In dealing with questions in space it is to be remarked first that instead of two axes which had to be used in a plane to fix the position of a point, there are now three axes required, or rather three planes to measure from. Thus, if we want to denote the position of a point such as the corner of a table in a room, we may give its distances from two adjacent walls and from the floor. The three planes give planes to measure from, and their three intersections are three axes parallel to which the measurements are made. We will denote the axes by Ox , Oy , Oz .

Similarly, in dealing with the direction of a line in a plane, it was only necessary to give the angle that it made with a given axis Ox . When we are dealing with a direction in space it is necessary to give two angles to fix it. The common method of defining the direction will be seen from Fig. 213.

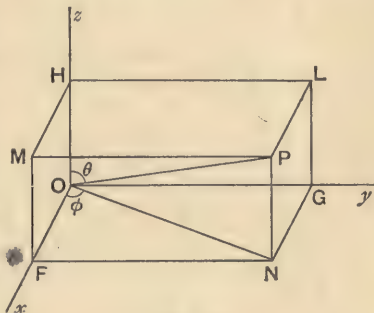


FIG. 213.

135. Let Ox , Oy , Oz be the three axes taken at right angles to one another. The position of P is denoted by its distances

from the planes yOz , zOx , xOy . These distances are PL , PM , PN , which we denote by x , y , z .

In the same figure the direction of a line such as OP can be fixed by the angles it makes with Ox , Oy , Oz . Another way of expressing the direction of OP is to give the angle that ON makes with Ox , and also the angle OP makes with ON , PN being the perpendicular from P to the plane xOy .

For example, in astronomy, if OP is the direction of a star seen by an observer at O , and if

Ox is the north line,

Oy towards the west,

Oz vertical,

xON is called the azimuth, or bearing of the star,

and PON is called its altitude.

In magnetism, if OP is the direction of a magnet freely suspended from its centre of mass,

xON is the declination,

and PON is the dip.

From the figure,

$$PL^2 + PM^2 = PH^2;$$

$$\therefore PL^2 + PM^2 + PN^2 = PH^2 + PN^2 = OP^2.$$

136. If OP represents a force in direction and magnitude, we can first resolve it into resolutes represented by OH and ON , and then resolve ON into OF and OG . So that OF , OG , OH are the three resolutes of OP along the three axes.

Now these can be expressed with the letters of the figure as follows :

$$ON = PH = OP \sin \theta,$$

$$OF = ON \cos \phi = OP \sin \theta \cos \phi,$$

$$OG = ON \sin \phi = OP \sin \theta \sin \phi,$$

$$OH = OP \cos \theta.$$

Again, if α, β, γ are the angles OP makes with Ox, Oy, Oz , since PF is perpendicular to OF, we have

$$OF = OP \cos \alpha ;$$

similarly,

$$OG = OP \cos \beta,$$

$$OH = OP \cos \gamma,$$

and we have the result, since

$$OP^2 = OF^2 + OG^2 + OH^2,$$

$$\therefore \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1,$$

so that α, β, γ are not all independent of one another.

α, β, γ may be called the **Direction Angles** of OP, and $\cos \alpha, \cos \beta, \cos \gamma$ are called the **Direction Cosines** of OP.

We have thus seen how to replace a force by three resolutes along three axes.

If we have any number of forces like P meeting in a point, we can replace each by its resolutes along the three axes, and then add the resolutes along each axis together, and we will thus get the resolute of the resultant. Thus, if the resultant is R and its resolutes are X, Y, Z,

$$X = \sum P \cos \alpha,$$

$$Y = \sum P \cos \beta,$$

$$Z = \sum P \cos \gamma,$$

$$R^2 = X^2 + Y^2 + Z^2,$$

and if R makes angles a, b, c with the axes,

$$\cos a = X/R,$$

$$\cos b = Y/R,$$

$$\cos c = Z/R.$$

If the forces are in equilibrium,

$$R = 0 ;$$

$$\therefore X^2 + Y^2 + Z^2 = 0 ;$$

$$\therefore X = 0, \quad Y = 0, \quad Z = 0.$$

These three conditions are necessary and sufficient for equilibrium.

Example. Prove that if a number of forces meet in a point and the sums of the resolutes in three different directions not in one plane, but not necessarily at right angles, separately vanish, the forces are in equilibrium.

137. We will generally resolve forces as here in three directions at right angles, but it is possible to resolve a force into three components in any given non-coplanar directions. Thus, in Fig. 206, Ox , Oy , Oz are supposed not to be at right angles,

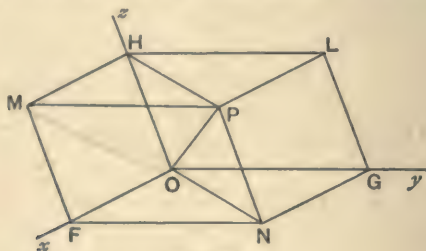


FIG. 214.

and we can still, by simply following the parallelogram of forces, see that OP can be replaced by components OF , OG , OH , where OP is the diagonal of the parallelepiped of which OF , OG , OH are conterminous edges, but the algebraical expressions for these components would be much more complicated.

In many cases a question on forces in space may easily be reduced to one of forces in a plane by compounding two of the forces into one.

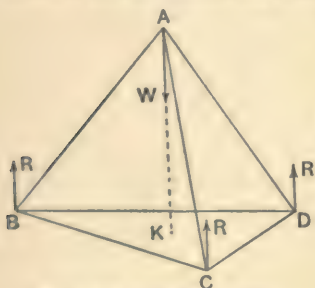


FIG. 215.—External forces.

Examples. 1. A tripod consists of three equal weightless rods, each of length a , jointed together at the upper ends, where a weight W is suspended and resting on a smooth horizontal plane. The tripod is kept from slipping by a rope of length $3a$ passing round the feet. Find the tension of the rope and the stress in the rods.

As the rods are weightless, the stresses act along the rods.

Let

S be the stress in each rod,

T be the tension of the rope.

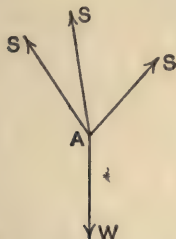


FIG. 216.—Forces at A.

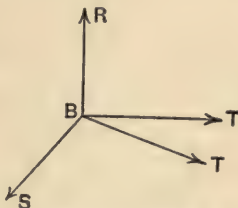


FIG. 217.—Forces at B.

The figure forms a regular tetrahedron of edge a , and by its geometry,

$$BK = a/\sqrt{3};$$

$$\therefore AK = a\sqrt{2}/\sqrt{3};$$

$$\therefore \cos BAK = \sqrt{2}/3.$$

Therefore at A we have three forces each S , and each making an angle $\theta = \cos^{-1}\sqrt{2}/3$ with the vertical; therefore resolving vertically for the joint A,

$$3S\sqrt{2}/3 = W;$$

$$\therefore S = W/\sqrt{6}.$$

At B, horizontally along BK,

$$S \sin \theta = 2T \cos 30^\circ;$$

$$\therefore \frac{S}{\sqrt{3}} = 2T \frac{\sqrt{3}}{2};$$

$$\therefore S = 3T;$$

$$\therefore T = \frac{W}{3\sqrt{6}}.$$

2. A body of weight 38 lbs. is supported by three strings from points A, B, C in the same horizontal plane. The body is vertically below a point O in the plane, and $OA = 10$ ft., $OB = 18$, $OC = 32$. If OB and OC are at right angles while OA makes an angle $\tan^{-1}\frac{3}{4}$ with CO produced, prove that the tensions of the strings are 26, 10, 10 lbs. wt. respectively.

(Hint. Think of each tension resolved into two components, one vertical and the other in the horizontal plane; the vertical

components balance the weight, those in the horizontal plane must be in equilibrium.)

3. OA, OB, OC are axes mutually at right angles. P is equally distant from the three planes OAB, etc. PL, PM, PN are the perpendiculars from P on the three axes, and PR is the perpendicular on the plane AOB. If PL, PM, PR, LR, MR, NR are weightless bars attaching P to fixed points L, M, N, find the stresses in the bars when a force F acts at P parallel to OB.

4. OA, OB, OC are axes mutually at right angles. CA, CB, CO are weightless bars jointed together at C and to fixed points A, B, O. A force acts at C, whose components are X, Y, Z. * Find the stresses in the bars.

5. A weight W is supported at a point O by a tripod of three rods, OA, OB, OC, resting on the ground at A, B, C. The vertical through O meets the ground at N, and AN meets BC in D. Prove that the stress in OA is $W \cdot ND \cdot OA / AD \cdot ON$.

6. In the tripod of Example 5, $OA = 7$ ft., $OB = OC = 5$ ft., $ON = \sqrt{24}$ ft., and the angles ANB, BNC, CNA are all 120° . Calculate by the method of Example 5 the stresses in the bars if the weight supported is 100 lbs.

7. An electric wire, approximately horizontal, and having a tension of 100 lbs., passes round a post 20 ft. high, the two portions making an angle 150° with one another. A stay 25 ft. long is fixed to the top of the post and to the ground. What should be the tension of the stay so that there should be no horizontal stress on the post?

CHAPTER XIV.

ANY FORCES IN THREE DIMENSIONS.

138. To discover the conditions of equilibrium when a number of forces act at any points of a body in three dimensions, we shall follow the same method as before by simplifying the system of forces, making use of the results of Chapter IV.

We have, however, to examine first into the composition of couples in different planes. We saw in Chapter IV. that couples in the same or parallel planes compound together by simply adding their moments. Now we will have to deal with couples in different planes, and the question arises at once, "Can couples in different planes be compounded together in any way?"

We shall be able to shew that we can represent couples by means of straight lines as we did forces. For the plane in which a couple lies may be sufficiently indicated by a straight line perpendicular to the plane, for it does not matter whether the couple is in a given plane or in any parallel one. Also the magnitude of the moment of the couple, which, it will be remembered, is the only other thing of consequence about the couple, may be represented by the length of this line, so that we have a line of a certain length and direction completely representing the action of the couple. The couple is thus a vector quantity, and may be expected to obey the vector laws.

There is, however, an important difference between the vector of a force and the vector of a couple. In the case of a force the vector may be thought of as placed anywhere along the line of action; in other words, one end (or the origin) of the vector may be taken anywhere along a certain straight line. In the case of a couple the origin may be taken anywhere in the plane of the couple, or in any parallel plane; in other words, at any point in space. For a couple may be moved anywhere in its own plane or into any parallel plane without altering its effect on a rigid body.

This statement again may be expressed by saying that while a force has direction, magnitude, and line of action, the couple has magnitude and line of action only.

The positive and negative directions of a couple in a plane will be distinguished by the vectors being drawn in opposite directions, and the direction of the vector is usually chosen so that the translation along the vector and rotation about it

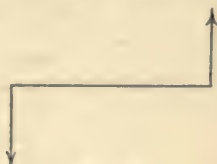


FIG. 218.

in the direction of the moment of the couple are connected with one another in the same way as the translation and rotation of an ordinary right-handed screw. Thus, for the couple shewn in Fig. 218, the vector should be drawn perpendicular to the paper, from the paper towards the reader.

The direction of the vector of a couple is often called its **Axis**.

If the plane of a couple is perpendicular to the line Ox , we can call it a couple whose vector is along Ox , or a couple about Ox .

139. Composition of Couples.

We will now prove that the vector law of composition applies to couples.

Let X, Y be the planes of two couples.

OO' the line of intersection of the planes.

xy their intersection with a plane perpendicular to both.

L, M the two couples, which may be represented by unit forces at distances L, M apart.

We can place the couples so that a force of each couple acts along OO' and in opposite directions, so that the two cancel.

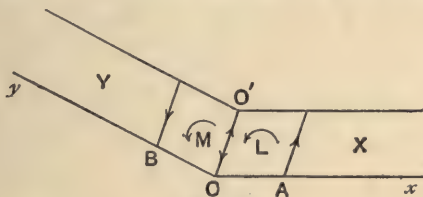


FIG. 219.

We are then left with two unit forces at A and B. In Fig. 220 these are perpendicular to the plane of the diagram, and being

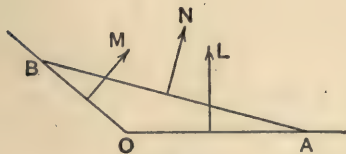


FIG. 220.



FIG. 221.

parallel form a couple whose force is unity and arm AB . Call this couple N . Now, drawing Fig. 221 with two sides parallel

and equal to L , M , we see that the triangle is congruent with AOB , its third side being perpendicular to AB , and therefore N is given in direction and magnitude by the ordinary vector law of composition.

140. Composition of a Number of Couples.

It follows that a couple can be resolved into components along three lines in space in exactly the same way as forces,

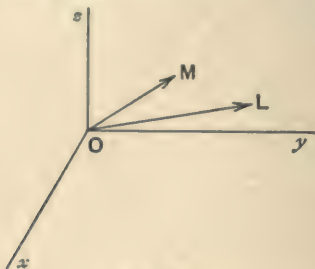


FIG. 222.

and that any number of couples in space can be compounded into one couple.

Thus, let L , M , N , ... be a number of couples, which may, by Art. 138, be represented by vectors all drawn through the same point O . The couple L can be resolved into components L_x , L_y , L_z , about three axes at right angles, and the components can then be added together so that we have finally three couples,

$$\Sigma L_x, \Sigma L_y, \Sigma L_z,$$

about the three axes, and these can be compounded into a single couple G , where

$$G^2 = (\Sigma L_x)^2 + (\Sigma L_y)^2 + (\Sigma L_z)^2,$$

and the direction of its vector is given by the angles α , β , γ it makes with the axes, where

$$\cos \alpha = (\Sigma L_x)/G,$$

$$\cos \beta = (\Sigma L_y)/G,$$

$$\cos \gamma = (\Sigma L_z)/G.$$

141. Composition of a Force and a Couple.

A single force and a couple cannot in general be compounded into either a single force or a couple. For example, if we have two forces, each P , parallel to Ox forming a couple, and a force Q along Oz as in Fig. 223, we can compound the forces at O into a single force R in the plane xz , but we cannot compound this with the remaining force P , because R and P neither meet in a point nor are parallel. In order to compound a force and a couple into a single force, they must be in the same plane; or in other words, their vectors must be perpendicular.

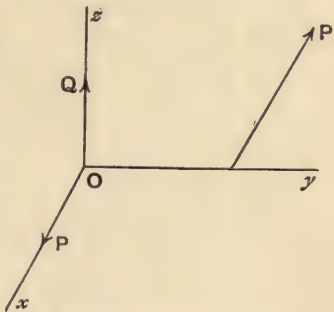


FIG. 223.—Force and couple.

But a force and a couple can be reduced to a force and a couple whose vectors coincide in direction. To do this, suppose we have a force R and a couple G , whose vector makes an angle θ with R . We can resolve G into two couples, one,

$$G \cos \theta \text{ about } R.$$

and the other,

$$G \sin \theta \text{ about a perpendicular to } R$$

Now the plane of the couple $G \sin \theta$ contains R , and this

couple and force can be compounded into a single force R parallel to the original. Hence we now have a force R and a couple $G \cos \theta$, whose vector is in the same direction as R , and as the vector of the couple may be drawn with any origin, we may say that these vectors coincide in position.

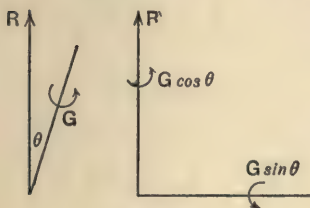


FIG. 224.

A force and couple whose vectors coincide in direction are said to form a **Wrench**, or a **Poinsot Wrench**.

142. Reduction of any System of Forces.

We can now see how to reduce any system of forces to its simplest form.

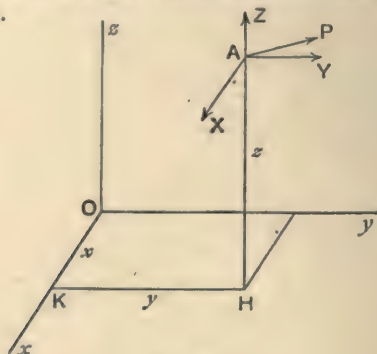


FIG. 225.

Let P be a force acting at a point A , whose coordinates are x, y, z . It can first be resolved into three resolutes X, Y, Z parallel to three axes Ox, Oy, Oz .

Let AH be the perpendicular from A on the plane xOy , and HK the perpendicular from H on Ox , so that

$$OK = x,$$

$$KH = y,$$

$$HA = z.$$

The force X at A can be replaced by

a force X at H

and a couple Xz , whose vector is parallel to Oy .

The last vector can be shifted to O .

Again, X at H can be replaced by

a force X at O

and a couple Xy , whose vector is in the negative direction of Oz .

Thus we have now replaced the original X by

	a force	X	at O	along	Ox ,
	a couple	Xz		about	Oy ,
and	a couple	$-Xy$		about	Oz .

Similarly, we can replace Y by

	a force	Y	at O	along	Oy .
	a couple	Yx		about	Oz ,
and	a couple	$-Yz$		about	Ox ,

and we can replace Z by

	a force	Z	at O	along	Oz ,
	a couple	Zy		about	Ox ,
and	a couple	$-Zx$		about	Oy .

Adding these together, the force P at A is equivalent to

	forces	X, Y, Z	at O	along the axes,
and	couples	$Zy - Yz, Xz - Zx, Yx - Xy$	about the axes.	

If we have any system of forces we can do the same with each, and so reduce the whole to

	forces	$\Sigma X, \Sigma Y, \Sigma Z$	along the axes,
and	couples	$\Sigma(Zy - Yz), \Sigma(Xz - Zx), \Sigma(Yx - Xy)$	about the axes.

The forces can now be compounded into a single force R at O , and the couples into one couple, G .

Thus, in any case a system of forces may be reduced to a single force at any assumed point and a couple. In general, these cannot be reduced further except to a Poinsot Wrench, but in special cases either the single force or the couple, or both, may vanish. Further, as in the case of forces in a plane, it will be seen that the force will be the same in magnitude and direction whatever point O is taken as base, but both magnitude and direction of the couple will vary with different bases.

The possibilities, therefore, for the reduction of a system of forces are :

- (i) A force and a couple.
- (ii) A force only.
- (iii) A couple only.
- (iv) Nothing at all, or equilibrium.

143. Conditions for Equilibrium.

It will be seen now that for equilibrium both the resultant force and the resultant couple must be zero, and these involve that the three components in each case are zero, and we therefore have six conditions,

$$\begin{aligned}\Sigma X &= 0, \\ \Sigma Y &= 0, \\ \Sigma Z &= 0, \\ \Sigma(Zy - Yz) &= 0, \\ \Sigma(Xz - Zx) &= 0, \\ \Sigma(Yx - Xy) &= 0.\end{aligned}$$

Examples. 1. Prove algebraically that if these equations are satisfied, and if a new point O' is chosen as origin instead of O , the six expressions there also will vanish.

2. Prove algebraically that if the forces reduce to a couple the components of the couple are the same for all base points.

144. Moment of a Force about a Line.

In dealing with forces in one plane we spoke only of the moment of a force about a point. It can be shewn (see *Elementary Dynamics*, Art. 180) that the moment is a measure of the tendency of the force to cause rotation about an axis through the point perpendicular to the plane containing the force and the point.

In dealing with forces in different directions in space it is necessary to examine the turning power of a force about any line in space. We can deduce satisfactory methods of measuring

the turning power, by using the fact that the effect of a force is the same as the sum of the effects of its components.

Suppose it is required to define the *Moment of a force about a line* in accordance with these ideas.

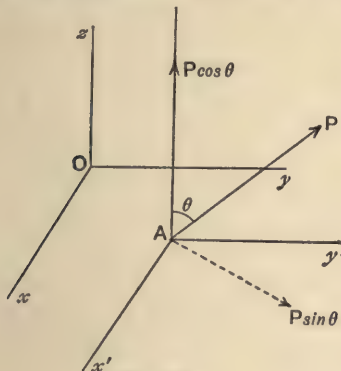


FIG. 226.

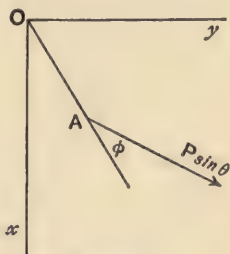


FIG. 227.

Let P be a force in any direction whose moment about Oz is required.

Let P meet the plane xOy in A .

Resolve P into components

$P \cos \theta$ at A parallel to Oz

and

$P \sin \theta$ at A perpendicular to Oz .

Now, $P \cos \theta$ being parallel to Oz , evidently has no tendency to turn a body about Oz , and the moment of P about Oz may therefore be defined as equal to the moment of $P \sin \theta$ about O .

If $P \sin \theta$ makes an angle ϕ with OA produced; or in other words, if OA makes an angle ϕ with a plane PAz' , parallel to the force and to the line about which the moment is wanted, and if

$$OA = a,$$

we can write the moment as

$$\begin{aligned} & P \sin \theta \cdot \sin \phi \cdot a \\ &= Pa \sin \theta \sin \phi. \end{aligned}$$

In the special case, when OA is the shortest distance between the force and the given line there is no component of P along OA , OA being perpendicular to P , and the moment can therefore be written in any case as $Pd \sin \theta$, where d is the shortest distance between the line and the force.

It will be found that the definition above is such that the moment is the same whatever point A is chosen as the point of application of the force:

This will be seen from Fig. 228, in which Ox is taken so that P is parallel to the plane zOx , and Oy is along the shortest distance between Oz and P , OA being the common perpendicular.

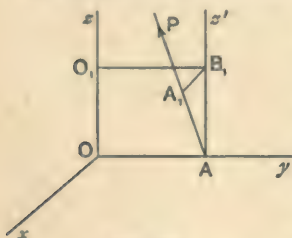


FIG. 228.

If, now, any other point A_1 is taken on P , and AB_1 is parallel to Oz , and A_1B_1 perpendicular to AB_1 and O_1B_1 perpendicular to Oz , it will be seen that whether we take the force as acting at A or A_1 , the component whose moment we have to take is $P \sin \theta$, and in one case it has to be multiplied by OA , and in the other by O_1B_1 , which equals OA . Hence the moment is independent of the position of A .

Summing up the results of this article in other words, we may put the definition of moment of a force about a line as follows :

If the force is perpendicular to the line its moment is the product of the force into its shortest distance from the line. If the force is not perpendicular to the line, it must be resolved into two components, one parallel and the other perpendicular to the line, and the moment is the product of the latter into its shortest distance from the line, which is also the shortest distance of the resultant force from the line.

It will further be seen that when P is replaced by $P \cos \theta$ and $P \sin \theta$, the latter may again be resolved into any two

convenient components, and the sum of the moments of these two will be equal to the moment of P about the given line.

145. Algebraical Expressions for the Moment of a Force about a Line.

Suppose with the previous notation the force P at $A(x, y, z)$ is resolved into three components X, Y, Z , parallel to the axes.

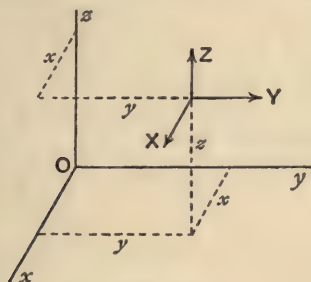


FIG. 229.

The moment about Oz

of X is $-Xy$,

of Y is Yx ,

and of Z is 0 .

Hence the total moment is

$$Yx - Xy,$$

and similarly the moments about Ox and Oy are

$$Zy - Yz$$

and $Xz - Zx$.

Hence the expressions

$$\Sigma(Zy - Yz), \quad \Sigma(Xz - Zx), \quad \Sigma(Yx - Xy),$$

found before for the couples about the axes, may also be

called the sums of the moments of the forces about the axes.

146. The conditions for equilibrium may therefore be expressed as follows :

A system of forces is in equilibrium if the sums of the resolved parts of the forces along any three lines at right angles are separately zero, and the sums of the moments of the forces about these lines are also separately zero.

It will readily be seen, however, that the three lines chosen for resolving along need not necessarily be the same as the three about which moments are taken, nor need either set, in fact, consist necessarily of lines at right angles to one another, but may be in any directions, provided they are not in one plane, but it is almost always most convenient to take them at right angles.

147. Resultant of Parallel Forces in Three Dimensions.

A number of parallel forces can be compounded two at a time, and will therefore always reduce either to a single force or a couple. If they reduce to a single force the resultant will be equal to their algebraic sum, and the moment of the resultant about any line will be equal to the sum of the moments of the separate forces.

Suppose the forces are $P_1, P_2, P_3, \dots$ parallel to Oz , and that they meet the plane xOy in points whose co-ordinates are $x_1, y_1; x_2, y_2, \dots$, and suppose the resultant R meets this plane in $\bar{x}, \bar{y}$.

Then $R = \Sigma P$.

Also taking moments about the line Oy ,

$$R\bar{x} = \Sigma Px;$$

$$\therefore \bar{x} = \frac{\Sigma Px}{\Sigma P}$$

exactly as in Art. 33 for forces in one plane.

Hence also, by the same reasoning as in Art. 107, the position of the centre of mass of a number of particles in space is given by

$$\bar{x} = \frac{\sum mx}{\sum m}, \text{ etc.},$$

as stated in that article.

148. Moment of a Couple about a Straight Line.

If a couple has a vector along Oz and a moment N , then its moment about a line OA , making an angle θ with Oz , is $N \cos \theta$.

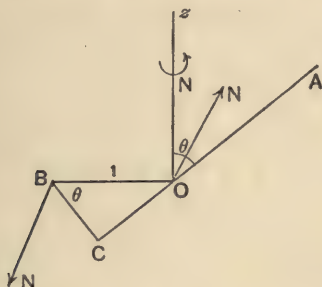


FIG. 230.

For the couple can be represented by two forces N at unit distance OB apart and perpendicular to the plane zOA , and with one of them passing through O . The moment of the other about OA will then be evidently $N \cdot BC$ or $N \cos \theta$, and therefore the total moment of the couple about OA is $N \cos \theta$.

A couple consequently has a moment about any line not at right angles to its vector; or in other words, not parallel to its plane.

149. Conditions of Equilibrium.

We have seen that a set of necessary and sufficient conditions of equilibrium, is that the sum of the resolved parts of the forces in each of three different non-coplanar directions

should separately vanish, and that the sums of the moments about three different non-coplanar lines should separately vanish.

It may be useful to examine this statement in another way.

It is proved in Art. 142 that a system of forces may reduce to either

- (i) A force and a couple.
- (ii) A force only.
- (iii) A couple only.
- (iv) Zero.

If the above six conditions hold, we see that the resultant force must be zero, for if there were a resultant (which it will be remembered is the same whatever base point is taken), it would require to be at right angles to each of the three directions, which is impossible, as the three are not in one plane. Hence the alternatives (i) and (ii) are impossible. We then see that the remaining conditions shew that (iii) is impossible, for if the sum of the moments of the forces about three lines are separately zero, and the forces reduced to a couple, this couple would have zero moment about each of the three lines, and these three lines would therefore have to be all parallel to the plane of the couple, which is contrary to the supposition that they are concurrent and non-coplanar. Hence the forces must be in equilibrium.

It will also readily be seen that the six conditions are necessary. For if five only of these were given, the forces might reduce to a single force or a couple according to what five conditions held.

Thus, if the sums of the resolutes along Ox and Oy are zero, and the sums of the moments about the three lines are zero, so that we have five conditions only, all these conditions would be satisfied if the forces reduced to a single resultant along Oz . Again, if the sums of the resolutes along the three axes are zero, and the sums of the moments about Ox and Oy zero,

the conditions would be satisfied if the forces reduced to a couple in the plane xOy .

150. Moment of a Force about a Point.

Moments may also be treated in a rather different way, which is especially useful for general investigations by means of vector notation.

The moment of a force about a point may be defined as a vector quantity, whose direction is perpendicular to the plane containing the force and the point, and whose magnitude is the product of the force into the perpendicular distance of the force from the point, and whose sense is connected with the direction of rotation produced by the force about the direction of the vector by the right-handed screw relation.

This would be exactly the sense in which the term moment about a point is used in the plane cases, if we treated the moment as a vector perpendicular to the plane of the diagram.

If AB represents the force in magnitude as well as in position, and ON is perpendicular to AB , then the moment about O is a vector perpendicular to the plane OAB , and equal to $P \times ON$, and therefore represented by double the area of the triangle OAB .

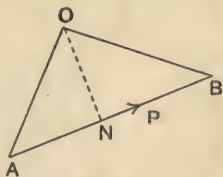


FIG. 231.

With this definition we can prove the following fundamental theorem :

If G is the vector moment of a force P about O , the moment about any line Oz through O is the resolved part of G along Oz .

Let AB represent the force P in magnitude as well as direction ; then its moment G about O is represented in magnitude by twice the area of the triangle OAB .

Let Oz make an angle θ with G . Draw a plane xOy through O perpendicular to Oz , and let CD be the projection of AB on this plane.

Then CD will be equal to the resolved part of P perpendicular to Oz , and therefore the moment of P about Oz will be represented by the product of CD , and the perpendicular from O

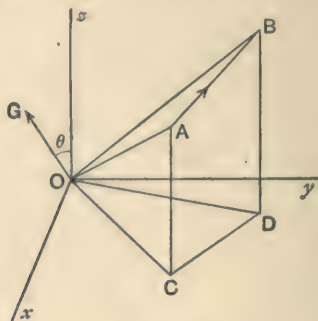


FIG. 232.

on CD , or, in other words, by twice the area of the triangle OCD .

Now Oz being perpendicular to OCD , and G being perpendicular to OAB , the angle between the planes OAB and OCD is θ .

$\therefore \triangle OCD$ being the projection of $\triangle OAB$,

$$\frac{\triangle OCD}{\triangle OAB} = \cos \theta,$$

or the moment about Oz is $G \cos \theta$.

151. Composition of Moments about a Point.

We can also prove the equally important result that *if two forces meet in a point, the vector sum of their moments about any point is equal to the moment of their resultant about the same point*, or, moments about a point of concurrent forces follow the vector law.

Let AB , AC represent the two forces, and O the point about which moments are taken. Construct the parallelopiped as in Fig. 233, and join AD , OG .

Then the moment of P about O is a vector perpendicular to $OABE$ and represented in magnitude by $OABE$. The moment of Q is perpendicular to $OACF$ and represented in magnitude

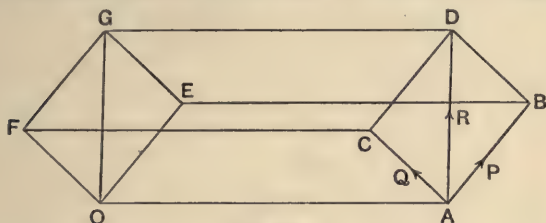


FIG. 233.

by $OACF$, or perpendicular to $EBDG$ and represented in magnitude by $EBDG$, while the moment of the resultant is perpendicular to $OADG$ and represented in magnitude by $OADG$.

It remains to shew that the vectors representing $OABE$ and $EBDG$ are equivalent to the vector representing $OADG$.

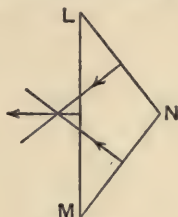


FIG. 234.

Draw a plane at right angles to the edges OA , etc., of the triangular prism $ABDOEG$, and let the section be LMN . The areas $OABE$, $EBDG$, $OADG$, are respectively equal to MN , NL , LM , each multiplied by the length of the prism.

Therefore the vector moments of the forces P , Q , R are proportional to MN , NL , LM , and they are also perpendicular to them, and being parallel to the plane of section can be treated as in that plane. Consequently the moments follow the usual vector law, for two vectors perpendicular and proportional to

two sides of a triangle are equivalent to a vector perpendicular and in the same proportion to the third side.

Examples. 1. Bifilar Suspension. A rod of weight W hangs at rest suspended horizontally by two vertical threads, each of length l and attached at distances a from the centre of the rod. Find the couple necessary to hold it when making an angle θ with its equilibrium position.

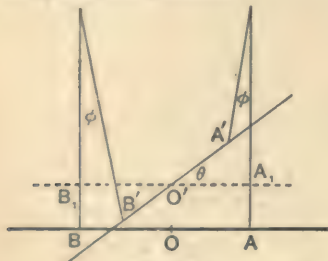


FIG. 235.

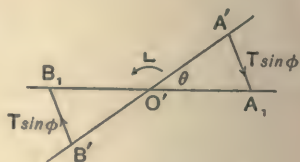


FIG. 236.

Let AB be the equilibrium position.

When the rod is turned round its centre O rises, so that it comes into a position O' , $A'B'$ being the new position of AB , and $A'B'$ still lying in a horizontal plane.

Fig. 236 represents a plan (looking down from above), A_1B_1 being the points in the horizontal plane of $A'B'$, which are vertically above AB .

Let ϕ = the angle each string makes with the vertical in the displaced position.
 T = the tension of each string.
 L = the required couple.

Geometrically we have

$$A_1A' = l \sin \phi = 2a \sin \frac{\theta}{2}; \dots\dots\dots(1)$$

also resolving vertically,

$$2T \cos \phi = W. \dots\dots\dots(2)$$

The horizontal component of T is $T \sin \phi$ acting along $A'A_1$. Hence taking moments about the vertical through O ,

$$\begin{aligned} L &= 2 \cdot T \sin \phi \cdot a \cos \frac{\theta}{2} \dots\dots\dots(3) \\ &= 2Ta \sin \phi \cos \frac{\theta}{2}; \end{aligned}$$

from (2) and (3),
$$\frac{L}{W} = a \tan \phi \cos \frac{\theta}{2}.$$

Now
$$\sin \phi = \frac{2a}{l} \sin \frac{\theta}{2};$$

$$\therefore \tan \phi = \frac{2a \sin \frac{\theta}{2}}{\sqrt{l^2 - 4a^2 \sin^2 \frac{\theta}{2}}},$$

$$\therefore \frac{L}{W} = \frac{a^2 \sin \theta}{\sqrt{l^2 - 4a^2 \sin^2 \frac{\theta}{2}}}.$$

2. A square trap-door ABCD, capable of turning about smooth hinges AB, in a horizontal line, is held at an angle α with the vertical by a rope attached at C to a point E vertically above B. If the weight of the door is W and $BE = BC = 2a$, find the tension of the string and the reactions at the hinges.

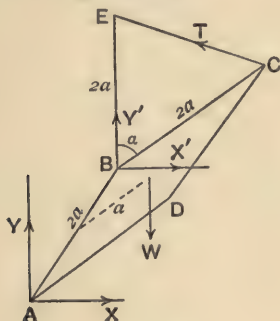


FIG. 237.

Let X, Y, X', Y' be the components of the reactions at A and B in the horizontal and vertical directions as marked. There will be no component along AB.

Resolving horizontally,
$$X + X' = T \cos \frac{\alpha}{2}, \dots\dots\dots(1)$$

vertically,
$$Y + Y' + T \sin \frac{\alpha}{2} = W, \dots\dots\dots(2)$$

moments about AB,

$$T2a \cos \frac{\alpha}{2} = Wa \sin \alpha, \dots\dots\dots(3)$$

moments about BE,

$$X2a = 0, \dots\dots\dots(4)$$

moments about line of action of X',

$$Y2a = Wa. \dots\dots\dots(5)$$

Hence from (4) $X = 0$.

$$\text{„ (5) } Y = \frac{W}{2}.$$

$$\text{„ (3) } T = W \sin \frac{\alpha}{2}.$$

$$\text{„ (1) } X' = T \cos \frac{\alpha}{2} = \frac{W}{2} \sin \alpha.$$

$$\text{„ (2) } Y' = W - W \sin^2 \frac{\alpha}{2} - \frac{W}{2} = \frac{W}{2} \cos \alpha.$$

3. Suppose in the last example the rope is attached to D instead of C. Find the reactions and tension when the door is horizontal.

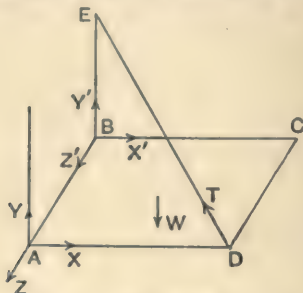


FIG. 238.

In this case T has a component parallel to AB , and there will therefore be a component of the reactions in the line of the hinges; let these be Z, Z' .

The components of T in each of the three directions is $\frac{T}{\sqrt{3}}$, for DE is the diagonal of a cube of which BA, BC, BE are conterminous edges, and therefore DE makes equal angles with each edge and the cosine of this angle is $\frac{BE}{DE} = \frac{1}{\sqrt{3}}$.

Hence the equations are:

$$\text{Resolving vertically, } Y + Y' + \frac{T}{\sqrt{3}} = W, \dots\dots\dots(1)$$

$$\text{Resolving parallel to BC,} \quad X + X' = \frac{T}{\sqrt{3}}, \quad \dots\dots\dots(2)$$

$$\text{parallel to AB,} \quad Z + Z' = \frac{T}{\sqrt{3}}, \quad \dots\dots\dots(3)$$

$$\text{moments about AB,} \quad \frac{T}{\sqrt{3}} 2a = Wa, \quad \dots\dots\dots(4)$$

$$\text{moments about BC,} \quad Y 2a + \frac{T}{\sqrt{3}} 2a = Wa, \quad \dots\dots\dots(5)$$

$$\text{moments about vertical through B,} \quad X = 0. \quad \dots\dots\dots(6)$$

$$\text{Hence from (4)} \quad T = \frac{W\sqrt{3}}{2},$$

$$\text{,, (6)} \quad X = 0,$$

$$\text{,, (2)} \quad X' = \frac{T}{\sqrt{3}} = \frac{W}{2},$$

$$\text{,, (5)} \quad Y = \frac{W}{2} - \frac{T}{\sqrt{3}} = 0,$$

$$\text{,, (1)} \quad Y' = W - \frac{T}{\sqrt{3}} = \frac{W}{2},$$

$$\text{,, (3)} \quad Z + Z' = \frac{W}{2}.$$

As Z and Z' act along the same straight line they will come in, in the form $Z + Z'$, into any equation in which they appear at all, and hence all we can obtain with regard to these components of the reaction is that their sum is $\frac{W}{2}$, but there is no means of telling how much of this will be borne by one hinge, and how much by the other ; it depends on how the plate has been placed in position, and a small knock may alter the division of the reaction between the two.

Such reactions are said to be **Indeterminate**.

4. A circular disk, of weight W and radius r , is suspended horizontally by three equal vertical strings attached at equal distances apart to its circumference. A given horizontal couple L being applied to the disk, find the equilibrium position and the tension of the strings.

5. A square table of weight W stands on three legs at three of its corners A, B, C . If the centre of mass is at distances x, y from AB and BC , find the pressures of each leg on the ground.

6. The line of hinges of a door makes an angle θ with the vertical. Find the couple required to hold it open at an angle ϕ with the equilibrium position.

7. A picture is suspended in the usual way by a string attached symmetrically to two points A and B of the back, and slung over a nail C in a smooth wall. Shew that if the perpendicular CM from C to AB is of length l and makes an angle ϕ with the wall, the distance of the centre of mass of the picture from the wall is

$$l \sin \phi + \tan \phi \sqrt{k^2 - l^2 \sin^2 \phi},$$

where k is the distance from M to the line of contact with the wall.

8. A heavy square lamina is suspended by two hinges at the ends of a horizontal edge, and is kept at an angle α with the vertical by a force F normal to the lamina and acting at a distance z from the horizontal line of the hinges and at a distance k from the middle line between the hinges. Find F and the stresses at the hinges.

9. A uniform rod AB, of weight W , rests with the ends A, B on a smooth horizontal floor, and against a smooth vertical wall respectively, the rod making an angle θ with the horizontal, and the vertical plane through it making an angle ϕ with the wall. Shew that horizontal forces $\frac{1}{2}W \cot \theta$ and $\frac{1}{2}W \cot \theta \cos \phi$, properly directed at A and B respectively, will keep the rod in equilibrium.

10. A rod AB, of length $2a$ and weight W , is hinged by a universal joint at A, and presses against a vertical wall at B at a distance d ($=2a \cos \alpha$) from A. If C is the foot of the perpendicular from A on the wall and BC makes an angle θ with the vertical, find the component reactions at the two ends of the rod, supposing the friction at B to be perpendicular to BC, and find the least coefficient of friction required to keep the rod thus.

11. If the frictional couple resisting the opening of the lid of a box is μ times the resultant force in the line of the hinges, shew that the lid will be about to open when the box is held so that the line of hinges makes an angle α with the vertical plane through the line of hinges where

$$b \sin \theta = \mu \cot \alpha,$$

$2b$ being the breadth of the lid.

12. A gasalier has three straight branches equally spaced round the descending tube and perpendicular to it. It is hanging out of plumb, and to make the tube vertical a horizontal force P applied at the bottom and making an angle θ with one of the branches is required. Find the weights which, placed at the ends of two of the branches, will effect the same object.

13. Reduce to their simplest form (single force, couple, or Poinset wrench) the systems of forces in the following examples, ABCDEFGH being a cube :

- (i) P along DE , and P along BG .
- (ii) P along each of GF , FE , EH , BC , CD , DA .
- (iii) P along each of GE , ED , DB , BG .

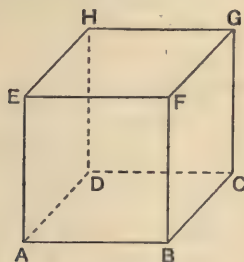


FIG. 239.

CHAPTER XV.

VECTORS IN SPACE.

152. In my *Elementary Dynamics* are given some of the elements of vector notation and vector algebra, including the definition of the Scalar Product of two vectors. The results given there are summarised here, and reference may be made to that book for fuller explanations and proofs.

If $\mathbf{A}$, $\mathbf{B}$ are two vectors, the scalar product of $\mathbf{A}$ and $\mathbf{B}$ is defined as the scalar quantity $\underline{AB \cos \theta}$, where A , B are the magnitudes of $\mathbf{A}$ and $\mathbf{B}$, and θ is the angle between them. The scalar product is denoted by $\mathbf{A} \cdot \mathbf{B}$, and read $\mathbf{A}$ dot $\mathbf{B}$.

We then have the following results in vector algebra :

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A},$$

$$\mathbf{A} - \mathbf{B} = -(\mathbf{B} - \mathbf{A}),$$

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}), \text{ etc.},$$

$$m(\mathbf{A} + \mathbf{B}) = m\mathbf{A} + m\mathbf{B}, \quad \text{if } m \text{ is any scalar,}$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A},$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}.$$

In dealing with questions relating to moments it is convenient to have a notation for the vector which represents the moment of a force about a point. We therefore introduce the **Vector Product** of two vectors, and define it thus :

The Vector Product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is a vector perpendicular to both, whose magnitude is equal to the product

$$\underline{AB \sin \theta},$$

where A , B are their magnitudes and θ is the angle between them.

We will use Gibbs's notation $\mathbf{A} \times \mathbf{B}$ for this vector, and it should be read $\mathbf{A}$ cross $\mathbf{B}$.

The direction of $\mathbf{A} \times \mathbf{B}$ must be such that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{A} \times \mathbf{B}$ form a right-handed system of axes, that is to say, the direction of $\mathbf{A} \times \mathbf{B}$ is the direction of translation of a right-handed screw

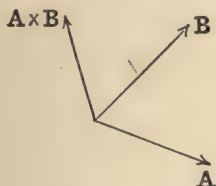


FIG. 240.

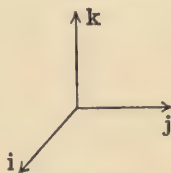


FIG. 241.

whose rotation would carry $\mathbf{A}$ towards $\mathbf{B}$ through an angle of less than 180° .

We see at once that $\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}$.

Thus the vector product depends on the order of the factors.

As a simple special case, suppose $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors (that is, vectors of unit magnitude) along three rectangular axes.

Then it will be seen at once (Fig. 241) that

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k},$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i},$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$

Also, as regards the scalar products, we have

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1,$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{i} = \mathbf{i} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = 0.$$

In the case of the moment of a force about a point, if $\mathbf{P}$ is the vector of the force and $\mathbf{r}$ the vector of a point on the line of application, then the moment about the origin is

$$\mathbf{r} \times \mathbf{P}.$$

153. The proof that we have given in Art. 151 that the vector sum of the moments of two concurrent forces is equal to the moment of the resultant applies to any two vectors. It is expressed by saying that

$$\mathbf{r} \times \mathbf{P} + \mathbf{r} \times \mathbf{Q} = \mathbf{r} \times (\mathbf{P} + \mathbf{Q}),$$

or in general, if $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are any three vectors

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C},$$

and likewise $(\mathbf{B} + \mathbf{C}) \times \mathbf{A} = \mathbf{B} \times \mathbf{A} + \mathbf{C} \times \mathbf{A}.$

Notice that from the definition we have necessarily

$$\mathbf{A} \times \mathbf{A} = 0.$$

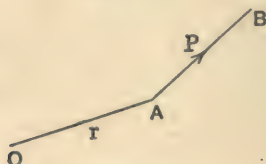


FIG. 242.

In Fig. 242, if B is any point on the line of action of $\mathbf{P}$, the vector AB, being in the same direction as $\mathbf{P}$, is equal to $\mathbf{P}$ multiplied by some scalar, say $\mathbf{AB} = c\mathbf{P}$, c being a scalar; then if $\mathbf{r}'$ is the vector to B from O,

$$\mathbf{r}' = \mathbf{r} + c\mathbf{P},$$

and if we suppose P to act at B instead of at A the moment about O

$$\begin{aligned} &= \mathbf{r}' \times \mathbf{P} = (\mathbf{r} + c\mathbf{P}) \times \mathbf{P} \\ &= \mathbf{r} \times \mathbf{P} + c\mathbf{P} \times \mathbf{P} \\ &= \mathbf{r} \times \mathbf{P}, \end{aligned}$$

since $\mathbf{P} \times \mathbf{P} = 0$. Hence it does not matter at what point of the line of action we think of $\mathbf{P}$ as acting, we get the same result for the moment about O.

154. Any vector can be expressed in terms of three vectors which are not coplanar. For the vector can be resolved into

components in any three directions, and hence if $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are three non-coplanar vectors, any other vector $\mathbf{P}$ can be resolved along them, and we can write

$$\mathbf{P} = a\mathbf{A} + b\mathbf{B} + c\mathbf{C},$$

where a , b , c are scalars or simply numbers.

In particular, if $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors at right angles a vector $\mathbf{P}$ can be written

$$\mathbf{P} = P_1\mathbf{i} + P_2\mathbf{j} + P_3\mathbf{k},$$

where P_1 , P_2 , P_3 are the resolved parts of $\mathbf{P}$ in the three directions.

In the same way, any vector in the plane of $\mathbf{A}$ and $\mathbf{B}$ can be written in the form

$$a\mathbf{A} + b\mathbf{B},$$

and conversely a vector of the form $a\mathbf{A} + b\mathbf{B}$ is in the plane containing $\mathbf{A}$ and $\mathbf{B}$.

Examples. By writing

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$$

and similar notation for other vectors, prove that

$$(i) \quad \mathbf{A} \cdot \mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3.$$

$$(ii) \quad \mathbf{A} \times \mathbf{B} = (A_2B_3 - A_3B_2)\mathbf{i} + (A_3B_1 - A_1B_2)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k}.$$

$$(iii) \quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.$$

All three of these results are extremely important, and the student should not proceed further before proving them. He must also see clearly what is meant by each group of symbols.

155. In connection with three vectors there are several important products.

Thus $(\mathbf{A} \cdot \mathbf{B})\mathbf{C}$ will be understood at once as the vector $\mathbf{C}$ multiplied by a scalar, and is therefore a vector in the same direction as $\mathbf{C}$.

$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ is a scalar quantity in which first the vector $\mathbf{A} \times \mathbf{B}$ is taken, and then the scalar product of it and the vector $\mathbf{C}$ formed.

Since $\mathbf{A} \times \mathbf{B}$ is a vector perpendicular to $\mathbf{A}$ and $\mathbf{B}$, and representing the parallelogram ODEF in Fig. 243 in magnitude, the scalar product of $\mathbf{A} \times \mathbf{B}$ and $\mathbf{C}$ will be got by multiplying this

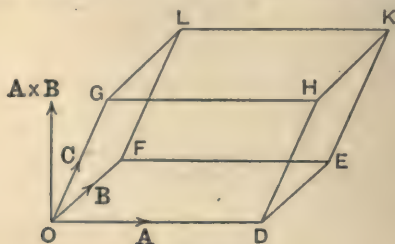


FIG. 243.

parallelogram by the resolved part of $\mathbf{C}$, perpendicular to the parallelogram. The result is therefore the volume of the parallelepiped whose edges are $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$.

Hence we have also

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B},$$

each of these representing the volume of the same parallelepiped.

The vector product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ will be a vector perpendicular to $\mathbf{B} \times \mathbf{C}$, and hence in the same plane as $\mathbf{B}$ and $\mathbf{C}$; consequently it follows at once that it can be expressed in the form $b\mathbf{B} + c\mathbf{C}$, as is obtained for it in example (iii) above.

Example. Verify by the method of the Examples of Art. 154 that

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A}.$$

156. Any Forces.

Returning now to any system of forces such as $\mathbf{P}$ acting at a point $\mathbf{A}$ whose vector is $\mathbf{r}$, we see first that the force $\mathbf{P}$ at $\mathbf{A}$ is equivalent to $\mathbf{P}$ at the origin and a couple consisting of the original force $\mathbf{P}$ and an equal parallel and opposite force at $\mathbf{O}$. The moment of this couple expressed as a vector is $\mathbf{r} \times \mathbf{P}$ by Art. 150. Compounding, we get that the forces are equivalent to a single force $\Sigma \mathbf{P}$ and a couple $\Sigma \mathbf{r} \times \mathbf{P}$.

The sums are of course vector sums.

We will denote these by

$$\mathbf{R} = \Sigma \mathbf{P}$$

and

$$\mathbf{G} = \Sigma \mathbf{r} \times \mathbf{P}.$$

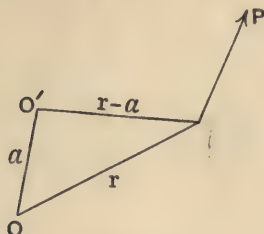


FIG. 244.

Now, if the moment is wanted about another point O' whose vector is $\mathbf{p}$, we have it at once as

$$\begin{aligned} \mathbf{G}' &= \Sigma (\mathbf{r} - \mathbf{p}) \times \mathbf{P} \\ &= \Sigma \mathbf{r} \times \mathbf{P} - \Sigma \mathbf{p} \times \mathbf{P} \\ &= \mathbf{G} - \mathbf{p} \times \Sigma \mathbf{P} \\ &= \mathbf{G} - \mathbf{p} \times \mathbf{R}. \end{aligned} \dots\dots\dots (i)$$

157. We will prove next that for all origins $\mathbf{R} \cdot \mathbf{G}$ is constant. For $\mathbf{R}$ is the same for all origins, and

$$\begin{aligned} \mathbf{R} \cdot \mathbf{G}' - \mathbf{R} \cdot \mathbf{G} &= \mathbf{R} \cdot (\mathbf{G} - \mathbf{p} \times \mathbf{R}) - \mathbf{R} \cdot \mathbf{G} \\ &= -\mathbf{R} \cdot (\mathbf{p} \times \mathbf{R}) \\ &= 0, \end{aligned}$$

since $\mathbf{p} \times \mathbf{R}$ is perpendicular to $\mathbf{R}$.

Hence $\mathbf{R} \cdot \mathbf{G}$ is constant for all origins.

If ΣL , ΣM , ΣN are the component couples about three rectangular axes, and ΣX , ΣY , ΣZ the component forces along the same axes, this is, by Art. 154, Ex. (i), the same as saying that

$$\Sigma X \Sigma L + \Sigma Y \Sigma M + \Sigma Z \Sigma N$$

is constant for all origins.

If in (i) of Art. 156 $\mathbf{p}$ is parallel to $\mathbf{R}$,

$$\mathbf{p} \times \mathbf{R} = 0;$$

$$\therefore \mathbf{G}' = \mathbf{G},$$

or the resultant moment is the same at all points on a line parallel to the resultant force.

158. Next let us try to find a point at which the vector of the resultant moment is parallel to the resultant force. To do this draw a plane through O perpendicular to $\mathbf{R}$; then if $\mathbf{p}$ is the vector to a point A on this plane, and if the moment at A is parallel to $\mathbf{R}$, then

$$\mathbf{G} - \mathbf{p} \times \mathbf{R} \text{ is parallel to } \mathbf{R},$$

or

$$\mathbf{G} - \mathbf{p} \times \mathbf{R} = m\mathbf{R},$$

where m is some scalar.

Take the vector product of each side by $\mathbf{R}$;

$$\therefore \mathbf{R} \times \mathbf{G} - \mathbf{R} \times (\mathbf{p} \times \mathbf{R}) = \mathbf{R} \times m\mathbf{R};$$

also by Art. 154, Ex. (iii),

$$\begin{aligned} \mathbf{R} \times (\mathbf{p} \times \mathbf{R}) &= (\mathbf{R} \cdot \mathbf{R})\mathbf{p} - (\mathbf{R} \cdot \mathbf{p})\mathbf{R} \\ &= R^2\mathbf{p} - (\mathbf{R} \cdot \mathbf{p})\mathbf{R}; \end{aligned}$$

now

$$\mathbf{R} \cdot \mathbf{p} = 0, \text{ because } \mathbf{R} \text{ is perpendicular to } \mathbf{p};$$

$$\therefore \mathbf{R} \times (\mathbf{p} \times \mathbf{R}) = R^2\mathbf{p};$$

also

$$\mathbf{R} \times m\mathbf{R} = m(\mathbf{R} \times \mathbf{R}) = 0;$$

$$\therefore \mathbf{R} \times \mathbf{G} - R^2\mathbf{p} = 0;$$

$$\therefore \mathbf{p} = \frac{1}{R^2} \mathbf{R} \times \mathbf{G},$$

giving one point on the plane, since we have one definite value for $\mathbf{p}$. Hence there is one line such that at all points of it the vector of the resultant moment coincides with the direction of the resultant force.

This line is called **Poinsot's Central Axis**.

The moment about any point on this line is

$$\begin{aligned}
 & \mathbf{G} - \mathbf{p} \times \mathbf{R} \\
 &= \mathbf{G} - \frac{1}{R^2} (\mathbf{R} \times \mathbf{G}) \times \mathbf{R} \\
 &= \mathbf{G} + \frac{1}{R^2} \mathbf{R} \times (\mathbf{R} \times \mathbf{G}) \\
 &= \mathbf{G} + \frac{1}{R^2} \{ -(\mathbf{R} \cdot \mathbf{R})\mathbf{G} + (\mathbf{R} \cdot \mathbf{G})\mathbf{R} \} \\
 &= \mathbf{G} + \frac{1}{R^2} \{ -R^2\mathbf{G} + (\mathbf{R} \cdot \mathbf{G})\mathbf{R} \} \\
 &= \frac{\mathbf{R} \cdot \mathbf{G}}{R^2} \mathbf{R},
 \end{aligned}$$

the numerical value of which is

$$\frac{RG \cos \theta}{R^2} R = G \cos \theta$$

if θ is the angle between $\mathbf{G}$ and $\mathbf{R}$.

The moment about any other point $\mathbf{p} + \mathbf{q}$, where $\mathbf{p}$ still represents $\frac{1}{R^2} \mathbf{R} \times \mathbf{G}$, is

$$\begin{aligned}
 & \mathbf{G} - (\mathbf{p} + \mathbf{q}) \times \mathbf{R} \\
 &= \frac{\mathbf{R} \cdot \mathbf{G}}{R^2} \mathbf{R} - \mathbf{q} \times \mathbf{R},
 \end{aligned}$$

which is the resultant of two vectors $\frac{\mathbf{R} \cdot \mathbf{G}}{R^2} \mathbf{R}$ and $-\mathbf{q} \times \mathbf{R}$, which are at right angles, and therefore the numerical value is greater than that of $\frac{\mathbf{R} \cdot \mathbf{G}}{R^2} \mathbf{R}$.

In other words, the moment about any point on the central axis is less than the moment about any point not on it.

APPENDIX.

NOTE ON THE HISTORY OF STATICS.

THE first attempts at a discussion of Statical Principles are due to Archimedes (287-212 B.C.), who dealt with the principle of the lever, as it was long called ; in other words, with moments of parallel forces. He was thus led to the consideration of centres of mass, which he investigated to a remarkable extent, finding, for example, the centre of mass of a paraboloid.

Pappus, who gave the theorems of Art. 116, lived in the third century A.D.

Modern Statics may be said to begin with Stevinus (1548-1620), who based his work on the triangle of forces, but the systematic treatment in the modern way, developed from the parallelogram of forces, may be ascribed to Varignon (1644-1722).

The principle of Virtual Work (then called Virtual Velocities) is due to Lagrange (1736-1813), perhaps the greatest mathematician since Newton. Most of Lagrange's work in mechanics is based on this principle.

The theory of couples in general and the reduction of a system of forces in space to a wrench (Art. 141) is due to Poinsoet (1777-1859).

The theory of vectors originated with Hamilton (1805-1865) and Grassmann (1809-1877), and has been developed and simplified by many workers of the nineteenth century.

The general methods of graphical statics, including the funicular polygon, are due to Culmann (1821-1881).

ANSWERS.

Note.—Answers for forces are given in lbs. wt. unless otherwise stated.

CHAPTER II.

- p. 12.** 4. 5, 8·66. 5. 16, 12. 6. 11, 6, 23·1. 7. 7·28, 21·28.
 8. 1·44 ft., 10·44, 0·21 ft. 10. 12, 8·1. 12. 6·5
 13. 48° 12'. No.
- p. 13.** 14. $8, 4W/\sqrt{100-W^2}$. 15. 143° 8', 126° 52', 90°.
 16. 135° 35', 122° 53', 101° 32'.
- p. 15.** 1. 32·14, 38·30. 2. 6.
 3. 12·16, 25° 17' with the larger.
- p. 19.** 2. 7·21 at 73° 54' with the vertical.
 3. 3·32 at 33° 33' with the first.
- p. 20.** 4. 2·982 at 23° 38' with the force 4 towards 6.
 5. 3·95 at 14° 50' with force 5 towards 4.
 6. 1·55, 21·55. 7. 9·90, 12·04.
- p. 35.** 1. 8·66, 5, perpendicular to the string.
 2. 14·14, 14·14, perpendicular to the string.
 3. 1·25, 4·51, 16° 6' with the rod.
 4. 1·67, 4·41, 10° 54' with the rod.
 5. 60°, 30° with vertical, 8·66.
 6. 57·7 in., 9·61, 5·33, 53° 8'. 8. $W \sec \theta$.
- p. 36.** 9. $r(\sec a - 1)$, $W \tan a$, $W \sec a$.
 10. 6, 5·20. 11. $W \sin \beta / \sin (a + \beta)$, $W \sin a / \sin (a + \beta)$.
 13. 60°, 5·20, 3. 14. 67° 31'. 15. $\frac{2W}{\sqrt{3}}$, $\frac{W}{\sqrt{3}}$
 16. $2W \sin a$. 17. 67·32, 32·68. 18. 16, 4.
 19. $\frac{3}{5}W$. 20. 14·14. 21. 75·73. 22. 106·91.

- p. 37.** 24. 125·66. 25. $W \frac{\sqrt{r^2 - (k+r-h)^2}}{k+r-h}$, $W \frac{\sqrt{2rh-h^2}}{r-h}$, 114·56.
 28. $\mu(2P+W)$. 29. $W \sin \beta / \sin(a+\beta)$, $W \sin \alpha / \sin(a+\beta)$.
 31. 0·26*w*.
- p. 38.** 32. 69·5, 31·5. 35. 7·97, 12·81.
 36. 14° 29', 6·12, 23° 21' with vertical.
 37. 13·6, 6·61 oz. wt.
 38. 1·38. 39. 1·50, 1·91.
 40. 49·0, 81·2. 41. 1·58; 1·71, 3·29 ft.

CHAPTER III.

- p. 42.** (i) Force 20, distance 4 from larger to right, in all the others the force is to left of larger. (ii) 4, 20.
 (iii) -4, 20. (iv) 1, 110. (v) 0·1, 1190.
 (vi) 0·01, 11990.
- p. 43.** 1. (i) 16, 3 inches from first towards second.
 (ii) 12, 6·67. (iii) 9, 4·44.
 2. (i) 4, 12 inches from first away from second.
 (ii) 4, 20. (iii) 1, 400.
 3. $7\frac{1}{3}$, $2\frac{2}{3}$. 4. 50, 70. 5. 3, 18.
 6. 85. 7. $33\frac{1}{3}$, 4. 8. 43·75, 35.
- p. 50.** 3. (i) 1, 1 ft. beyond last. (ii) -1, 1 ft. before first.
 (iii) -1, opposite the 6.
 4. 3 lb.-wt.-ft. 5. $2\frac{1}{12}$, $1\frac{5}{12}$. 6. 54·1, 45·9. 7. 7·85, 8·15.
 8. 8·8 ft. from A. 9. 9 at B. 10. 100. 11. 3.
 12. 93·3, 74·7.
- p. 60.** 1. 51·52547, 1·00103; 51·52550, 1·00100. 2. 1 mm.
 3. $W'a+Wb/(a+b)$, 15·55. 4. 26·873.
- p. 61.** 7. 1 in., 9 lbs., 2 ins. from A.

CHAPTER IV.

- p. 77.** 1. 6, 10000 ft.-lbs. 2. 4·55, 80 ft.-lbs.
 3. 156·25, 6250 ft.-lbs. 4. $(W+W')/2$, $(W-W')/2$.
 5. 28, 112. 6. 37·3. 7. 8·3, 69·4 per cent.
- p. 78.** 8. 22·4, 760 ft.-lbs. 9. 57·9 per cent.
 11. 377, 7·86 ft.-lbs. 12. 0·30 in. 13. $(p \pm p')/2\pi b$.

CHAPTER V.

- p. 84.** 1. 23 lb.-wt.-ft. 2. 19.9. 3. 6.93.
- p. 86.** 1. 3 parallel to BC, 2.67 in. from BC.
2. Represented by BD but through C.
3. 6.67 from A.
- p. 90.** 6. (i) Force $5\sqrt{2}$ parallel OA, couple 15, 10, 13; 3 in. to left of A.
(ii) Force 5 at $53^\circ 8'$ with OA, couple 5, -10, 15; 2.75 to right of A.
(iii) $2P$ along AB.
7. 6.71, $26^\circ 34'$ with AB, meeting AB at distance $\frac{3}{5}AB$ from A.
8. $2Pa$ if a = side of square.
- p. 98.** 1. (i) 15, 15, 20, 25, $53^\circ 8'$ with horizontal.
(ii) 8.66, 8.66, 20, 21.79, $66^\circ 35'$.
2. 74.2, 53.9, $13^\circ 24'$ with horizontal.
3. $21^\circ 19'$.
- p. 99.** 4. $(W + 2W')a \sin \phi/b \sin (\theta + \phi)$,
 $(W + 2W')a \sin \theta \sin \phi/b \sin (\theta + \phi)$,
 $W + W' - (W + 2W')a \sin \phi \cos \theta/b \sin (\theta + \phi)$.
5. θ where $\tan \theta = \tan \phi \{1 - b(W + W')/a(W + 2W')\}$.
6. 28.7 cwt. 7. 21.8. 8. 93.75, 150.
9. $51^\circ 20'$. 10. 12.32 ft. 11. $38^\circ 40'$. 12. 0.414.
- p. 100.** 14. 50, 125, 43.3; 150. 15. 4.33, 7.5, 5.
18. $\sqrt{l^2 - d^2}/2\mu$ from the centre.
19. Peg must be within distance $\mu a/\sqrt{3}$ from the centre of a side, if $2a$ is length of side.
20. $b \tan \alpha/(a + b)$.
- p. 101.** 21. $\cos^{-1} \left(\frac{W + W'}{W'} \sin \alpha \right)$, $\tan \alpha$, horizontal.
22. 112.5 at each.
23. Components $\frac{W}{2} (\cos^2 \theta + \cos \theta \sin \theta)$,
 $W \left\{ 1 - \frac{\sin^2 \theta + \sin \theta \cos \theta}{2} \right\}$; $\frac{W}{2} (\cos \theta + \sin \theta)$.
25. 0.173, 129.9.

- p. 104.** 1. $Wl \sin(A+B)/b \sin A$, $\frac{Wl}{b} \sin(A+B)$, $W + T \cos A$.
 2. $W \cot \alpha$, $Wl(2 - \sin^2 \alpha)/(2l \sin \alpha - h)$.
 3. At A, $\left(P \frac{b}{a} - W \frac{a-b}{2a}\right) \frac{h}{\sqrt{a^2 - h^2}}$; at B, horizontal same as at A, vertical $P + W$.
 4. $wb^3/2c + Wd/c$. 5. $\frac{1}{2}$, $\frac{W}{2} \cos\left(\frac{\pi}{4} + \theta\right)$.
 7. $(r \cos \theta - l \sin \theta) \sin \theta / \{2r - (r \cos \theta - l \sin \theta) \cos \theta\}$.
- p. 105.** 9. Components parallel and perpendicular to AB, $W \sin \alpha$, $W(\cos \alpha + \frac{b}{d} \sin \alpha)$; $W \frac{b}{d} \sin \alpha$.
 12. $\frac{5}{8}$.
- p. 106.** 16. Pl/a . 17. L/a . 18. 185.8, 14.2, 40.

CHAPTER VI.

(Usually horizontal and vertical components of reactions are given.)

- p. 115.** 4. 8.66, 25; 8.66, 5; 8.66, 5.
 5. 6.50, 21.25; 6.50, 1.25; 6.50, 8.75.
 6. 8.4, 18.7; 8.4, 3.7; 8.4, 16.3.
 7. 8.4, 13.8; 8.4, 1.2; 8.4, 21.2.
 8. $\frac{W}{2} \cot \alpha$, $3 \frac{W}{2}$; $\frac{W}{2} \cot \alpha$, $\frac{W}{2}$.
 9. $W/(\tan \alpha - \tan \beta)$, $W(3 \tan \alpha - \tan \beta)/2(\tan \alpha - \tan \beta)$, etc.
 10. Resultant reaction at B, W ; at E, $W\sqrt{3}$.
 11. At B $\frac{W}{2}$, $3 \frac{W}{2} \cot \theta$; at E resultant $3 \frac{W}{2} \sec \theta$.
 12. $T \sin D$, $T \cos D + W + w$, if T = given thrust.
- p. 116.** 13. 7.22; 7.22, 2.5.
 14. 6.41; 5.92, 17.54.
 18. $W \left\{ 1 - \frac{a \cos \alpha \sin \theta}{2b \sin(\alpha + \theta)} \right\}$, $\frac{Wa \cos \alpha \cos \theta}{2b \sin(\alpha + \theta)}$.
 If $a \cos \alpha \sin \theta < b \sin(\alpha + \theta)$, μ must $> \cot \theta$;
 if $a \cos \alpha \sin \theta > b \sin(\alpha + \theta)$,
 μ must $> \frac{a \cos \alpha \cos \theta}{2b \sin(\alpha + \theta) - a \cos \alpha \sin \theta}$.

- p. 117.** 20. $\frac{5}{8}W \cot a, \frac{W}{8}$; $\frac{5}{8}W \cot a, \frac{5}{8}W$; $\frac{5}{8}W \cot a, \frac{7}{8}W$.
21. If $L > 4Wa$ the rods can remain horizontal;
 if $4Wa > L > Wa$, $\sin \theta_1 = \frac{1}{3} \left(\frac{L}{Wa} - 1 \right)$, $\theta_2 = 90^\circ$;
 if $L < Wa$, $\theta_1 = 0$, $\sin \theta_2 = L/Wa$,
 where θ_1, θ_2 are the angles BC, CD make with the vertical.
24. $3.33a, 2.87a$, if $2a$ is the length of each.
25. $W \cot a$.
26. $\frac{W}{2} \sqrt{1 + 9 \cot^2 a}$; $3 \frac{W}{2} \cot a, \frac{W}{2}$.
27. $\frac{2W \cos a \sin \beta \cos \lambda}{\sin(a - \beta - \lambda) - \sin \lambda} - W'$; if W = weight of a cylinder,
 W' = weight of wedge.

CHAPTER VII.

Note.—The stresses are given in the order in which the bars are marked in the diagrams. Tensions are negative.

- p. 130.** 1. 5.77, -11.55, -11.55, 11.55.
 2. 12.99, 47.63, -25.98, 2.89, -2.89, -31.75, 14.43.
- p. 131.** 3. 16.67, 13.33, 13.33, -23.57, 16.67, 4.71, 10, -18.86.
 4. 5.77, 11.55, 23.09, -2.89.
 5. 11, 8, 8, 5, 3, -4, 5, -9.53, -11.26.
 6. 18.38, 15.56, 10, 14.14, 16.97, 2.83, -2, 1.41, -3, 2.83,
 -13, -11, -12.
- p. 132.** 7. 6, 8, -4.8.
 8. 5.54, 7.29, -3.96.
 9. -30, -10, 14.14, -14.14, 14.14, 20.
 10. -15.28, -11.35, 13.55, 8.26.
 11. -20.71, -20.71, -20.71, 10.72, 10.72, 5.36.
- p. 133.** 12. -27.10, -17.69, 17.69, 21.70.
 13. -12.25, -10, 3.66, 17.32.
 14. -57.59, -10, 50.64, 25.71.
 15. At A, 25.98, 2; at C, 25.98, 18.
- p. 134.** 16. At A, 20.21, 40; at C, 20.21, 50.
 17. $W_1 \sin \theta / \sin B$.
 18. AD, BC must be parallel, and if a is the distance between them, the stress at each angle is parallel to AD and $= L/a$.

CHAPTER VIII.

- p. 145.** 1. $S = W' + \frac{W}{2l}(2l - x)$; $M = W'(2l - x) + \frac{W}{4l}(2l - x)^2$, where x is distance from fixed end.
2. Between end from which x is measured and nearest support, $S = Wx/2l$, $M = Wx^2/4l$;
between supports, $S = Wx/2l - W/2$;
 $M = Wx^2/4l - Wx/2 + \frac{W}{2}(l - a)$.
3. 4 ton-ft.
4. If x is measured from the end nearer the load, between this end and the load
 $S = 12 - \frac{3}{8}x$, $M = 12x - \frac{3}{10}x^2$,
beyond the load
 $S = 4 - \frac{3}{8}x$, $M = 40 + 4x - \frac{3}{10}x^2$;
greatest and least shearing stress 9 and 1 cwt. close to load; greatest bending moment 2.67 ton-ft., where $x = 6.67$ ft.
5. In the successive lengths of 4 ft. the shearing stresses are:
20 - x , 15 - x , 10 - x , 5 - x , - x ;
bending moments: $20x - \frac{x^2}{2}$, $20 + 15x - \frac{x^2}{2}$, $60 + 10x - \frac{x^2}{2}$,
 $120 + 5x - \frac{x^2}{2}$, $200 - \frac{x^2}{2}$.
 $S = 10$, 0 cwt.; $M = 82.5$, 110 cwt.-ft.
6. $12 - \frac{2}{3}x$, $96 - 12x + \frac{x^2}{3}$; at fixed end where $x = 0$.
7. 20.25, 20, 20 ton-ft.
- p. 146.** 10. $\frac{X}{2 \sin \theta} \left\{ \frac{2a - b}{a + b} \pm \frac{a}{a - b} \right\}$, where θ is the angle each brace makes with the vertical.

CHAPTER IX.

- p. 160.** 2. $h\sqrt{3}$, $h\sqrt{r+2}$.

CHAPTER X.

- p. 172.** 3. $0.523a$ if $2a$ is the side. 4. 2.5, 3.5.
5. 2.67, 1.94. 6. $\frac{a}{3}$ from the centre of the square.
7. $0.346a$ from centre of hexagon.

- p. 173.** 8. $0.267a$, $0.231a$ from centre.
 9. $0.408a$. 10. $0.321a$. 11. $h^2/(r+2h)$.
 12. Same as c.m. of tetrahedron. 13. As No. 12.
 14. $lh/2(l+r)$ from base if h =height and l =slant side.
 15. 2.33 ; 1.33 from the longer.
 16. $\pi r^2 a / (16a^2 - 2\pi r^2)$ from centre. 17. 0.494 from centre.
 18. $\frac{a}{4}$ from centre if $2a$ is side of square.
 19. $0.17a$ ($2a$ =side of square).
 20. On radius through 5 at distance $0.24 a$ from centre.
- p. 174.** 25. 3.71 in.
- p. 182.** 1. $\pi r^3(a - \frac{3}{8}r)/(6a^3 - \pi r^3)$ from centre.
 2. $\pi r^2 \left(\frac{a^2}{2} - \frac{h^2}{4} \right) / (8a^3 - \pi r^2 a + \frac{2}{3} \pi r^2 h)$ from centre.
 3. $\frac{h}{4} \frac{a^2 + 2ab + 3b^2}{a^2 + ab + b^2}$, from end a ;
 $\frac{h}{3} \frac{a + 2b}{a + b}$; $\frac{h}{3} \frac{l(a + 2b) + 3b^2}{l(a + b) + a^2 + b^2}$ if l is length of side.
 4. 238.9 cub. ft., 6.79 ft. from smaller end.
 7. $2a \sin^3 a / 3(a - \sin a \cos a)$ from centre of circle if the radii to the extreme points contain an angle $2a$.
 8. $\frac{h}{3}$ from base if h =height.
 9. $\frac{3}{4}(b+a) \frac{2r^2 - a^2 - b^2}{3r^2 - a^2 - ab - b^2}$ from centre of sphere, a, b being the distances of the bounding planes from the centre, and r the radius.
 10. $\frac{1}{3}(r-a)(2r^2 - ar - a^2)$; $\frac{3}{4} \frac{(r+a)^2}{2r+a}$.
- p. 183.** 16. $16a^2/3$; $3a$, $6a/5$; $64\pi a^3/5$, $10a/3$.
- p. 185.** 1. $0.547W$. 2. $2n$. 3. 45° . 7. $Wb/2a$, $W(\sqrt{a^2 + b^2} - a)$,

CHAPTER XI.

- p. 197.** 1. $W \cos \alpha / \sin \frac{\alpha}{2}$.
p. 198. 4. $2W$.

- p. 199.** 10. $F \cot \theta$. 11. Stable if $2b > a$.
 12. (i) Stable if $3b > 5a$. (ii) Stable if $8b > 3a$.
 13. Stable if (i) $4b > (3\pi - 4)a$. (ii) $3\pi b > (3\pi - 4)a$.
 (iii) $3\pi b > 4a$.
 14. Unstable. 15. Stable.

CHAPTER XIII.

- p. 214.** 3. $-F\sqrt{2}$, 0, F , F , F , $-F\sqrt{3}$.
 4. $X \cdot AC/OA$, $Y \cdot BC/OB$, $-Z - X \cdot OC/OA - Y \cdot OC/OB$.
 6. $13 \cdot 0$, $46 \cdot 4$, $46 \cdot 4$. 7. $86 \cdot 3$.

CHAPTER XIV.

- p. 235.** 4. Disk turns through an angle θ given by

$$L \sqrt{l^2 - 4a^2 \sin^2 \frac{\theta}{2}} = Wa^2 \sin \theta;$$

$$\text{tension} = W/3 \sqrt{1 - \frac{4a^2}{l^2} \sin^2 \frac{\theta}{2}}.$$
5. Wy/a , $W(a - x - y)/a$, Wx/a if a = side of square.
- p. 236.** 6. $Wa \sin \theta \sin \phi$ if $2a$ = breadth of door.
8. $F = \frac{Wa \sin \alpha}{h}$; components at hinges perpendicular to line of hinges; horizontal, $F \cos \alpha \frac{a - k}{2a}$, $F \cos \alpha \frac{a + k}{2a}$; vertical, $\frac{W}{2h} \{h + k \sin \alpha - a \sin^2 \alpha\}$, $\frac{W}{2h} \{h - k \sin \alpha - a \sin^2 \alpha\}$.
10. At A, $\frac{W}{2} \sin \theta \cos \theta$, $\frac{W}{2} \cot \alpha \cos \theta$, $W(1 - \frac{1}{2} \sin^2 \theta)$.
 At B, normal reaction = $\frac{W}{2} \cot \alpha \cos \theta$, friction = $\frac{W}{2} \sin \theta$.
 $\mu = \tan \alpha \tan \theta$.
12. $\frac{Pl}{a} \left(\cos \theta + \frac{\sin \theta}{\sqrt{3}} \right)$ and $\frac{Pl}{a} \left(\cos \theta - \frac{\sin \theta}{\sqrt{3}} \right)$, where l is the length of the vertical tube and a the length of each branch.
- p. 237.** 13. (i) $P\sqrt{2}$ through centre parallel to AE and a couple $2\sqrt{2}Pa$.
 (ii) $2P$ through centre parallel to EF.
 (iii) Couple $2\sqrt{2}Pa$ in a plane parallel to ABFE.

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